# RESPONSE OF NONLINEAR SYSTEMS TO UNUSUAL INPUTS

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# NAVAL POSTGRADUATE SCHOOL
## Monterey, California

# THESIS

RESPONSE OF NONLINEAR SYSTEMS
TO UNUSUAL INPUTS

by

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December 1974

Thesis Advisor: O.M. Baycura

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) Response of Nonlinear Systems to Unusual Inputs | | 5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; December 1974 |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) José António Joglar | | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940 | | 12. REPORT DATE December 1974 |
| | | 13. NUMBER OF PAGES 180 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Postgraduate School Monterey, California 93940 | | 15. SECURITY CLASS. (of this report) Unclassified |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |
| 16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. | | |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) | | |
| 18. SUPPLEMENTARY NOTES | | |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Nonlinear Systems | | |

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Handling nonlinear systems is not a simple task, especially if a computer is not available. Generally, although a precise result for every case would be preferable, an approximate solution is usually sufficient for most engineering solutions.

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The aim of this work is to test this technique in obtaining the solution of several time varying linear equations and to produce a computer program that improves the solution accuracy and ease of use.

Response of Nonlinear Systems
to Unusual Inputs

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Submitted in partial fulfillment of the
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MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
December 1974

## ABSTRACT

Handling nonlinear systems is not a simple task, especially if a computer is not available. Generally, although a precise result for every case would be preferable, an approximate solution is usually sufficient for most engineering solutions.

R. Boxer and S. Thaler developed a technique [1] presented in Proceedings of IRE that allow one to obtain the response of linear and nonlinear systems, without knowledge of the roots of the system characteristic equation.

The aim of this work is to test this technique in obtaining the solution of several time varying linear equations and to produce a computer program that improves the solution accuracy and ease of use.

TABLE OF CONTENTS

# I. INTRODUCTION

Two basic methods exist for solving the response of a system which generally is presented as an integro-differential equation: analytic methods and numerical methods.

For simple problems, analytic methods give precise solutions and have the advantage that these solutions can be given as a function of the parameters of the system.

As the order of the differential equation increases, numerical methods give less labor in providing a solution and can be, at some extent the only method available.

Laplace transform is limited to the case of linear differential equations. Nonlinear differential equations are out of the scope of this technique, preventing its application.

Even sometimes with linear systems, if the inverse integral is complex, it is difficult and sometimes impossible to find analytically the time function.

However, it is possible to approximate the Laplace transform of a linear system by a z-transform, obtaining thereafter a time series representing the values of the response at equally spaced intervals of time, by synthetic division.

The method [1] described and used in this paper utilizes the z-transform directly in order to obtain the solution of the linear and nonlinear systems.

In this paper, an explanation is given of this method applied to linear systems and its extension to time-varying and nonlinear systems.

A computer was used for faster and more reliable results, but the method described is compatible with handwork.

C.P.U. time in the computer, for this technique, is much smaller than other numerical methods due to the less number of iteration steps needed. This technique could reduce solution times by several orders of magnitude over conventional numerical methods.

## II. PRESENTATION OF THE METHOD

### A. PROCEDURE

The basic procedures of the method are given, before the description or the theory, to show its simplicity.

Suppose that one has the Laplace transform of an output response. It will be usually in the following form:

$$F(s) = \frac{a_o + a_1 s + a_2 s^2 + \cdots + a_k s^k + \cdots + a_n s^n}{b_o + b_1 s + b_2 s^2 + \cdots + b_k s^k + \cdots + b_m s^m} \qquad (1)$$

The steps to obtain the time response are:

1. Express the function F(s) as a rational fraction in powers of $s^{-1}$ by dividing the numerator and denominator by $s^m$.
2. Substitute for $s^{-k}$ a rational fraction in powers of $z^{-1}$ obtained from Table I and rearrange F(s) as a rational fraction in powers of $z^{-1}$.
3. Divide the resulting expression by T, where T is the time interval between points at which the result is desired.
4. Expand the fraction by synthetic division into a series of the form:

$$D_o + D_1 z^{-1} + D_2 z^{-2} + \cdots + D_n z^{-n} + \cdots \qquad (2)$$

where $D_n$, the coefficient of $z^{-1}$, is the approximate value of the time response at $t = nT$.

## B. LINEAR SYSTEMS

Take an arbitrary function f(t) as in Fig. 1(a). Its Laplace transform and inverse Laplace transform are defined as:

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \tag{3}$$

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} f(s)\, e^{st}\, ds \tag{4}$$

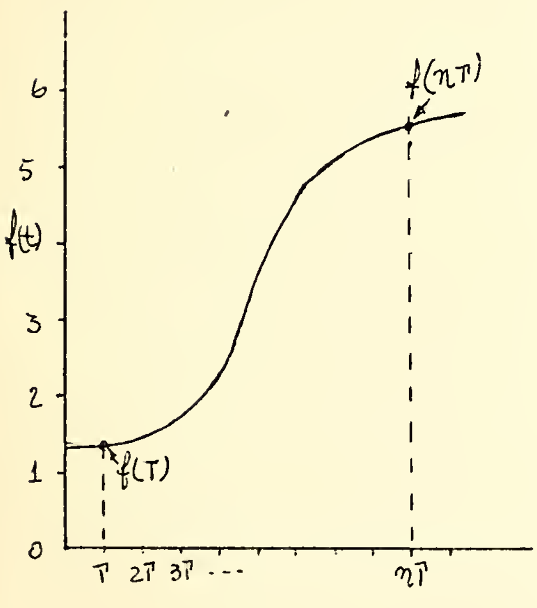



Fig. 1(a): Arbitrary f(t)

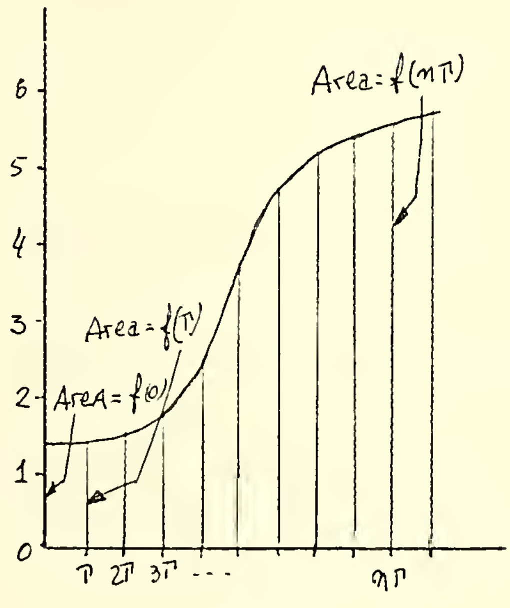



Fig. 1(b): Sampled f(t)

Figure 1(b) shows f(t) sampled by shifted unit impulses. The result is a series of impulses at t = nT.

Applying Laplace to the sampled function one gets:

$$F^*(e^{st}) = \sum_{n=0}^{\infty} f(nt)\, e^{-snT} \tag{5}$$

substituting in (5) $e^{sT}$ by z so:

$$z = e^{sT} \tag{6}$$

the result obtained is known as the z-transform:

$$F^*(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \tag{7}$$

and the inverse is given by:

$$f(nT) = \oint F^*(z)\, z^{n-1}\, dz \tag{8}$$

where the contour is a circle centered at the origin and including all the singularities of $F^*(z)$.

The equation (6) maps the $j\omega$ axis of the s-plane into a unit circle around the origin in the z-plane.

There are other methods for obtaining direct and inverse z-transforms.

In closed form the function will appear as:

$$F^*(z) = \frac{a_o + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}} \tag{9}$$

Division of the numerator by the denominator, will result in the coefficients of f(nT) as defined in (7), without the use of the inversion integral formula (8).

In order to apply this useful characteristic of z-transform theory to non-sampled systems one uses the inverse Laplace integral (4),

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

which can be expressed piecewise as:

$$f(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} f(s)e^{st}ds + \frac{1}{2\pi j} \int_{J\pi/T}^{J\infty} [F(s)e^{st} + f(-s)e^{-st}]ds \qquad (10)$$

The way the contour is divided is shown in Fig. 2. The first integral gives the part of the contour shown between $-(\pi/T) \leq \omega \leq \pi/T$ . The remaining contour $\omega > (\pi/T), \omega < -(\pi/T)$ is given by the second integral.

If 1/T is sufficiently large, a good approximation will be held by the first integral of (10) and the second integral may be discarded, being the resulting error given by:

$$e(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\infty} f(s)e^{st} + F(-s)e^{-st}\, ds \qquad (11)$$

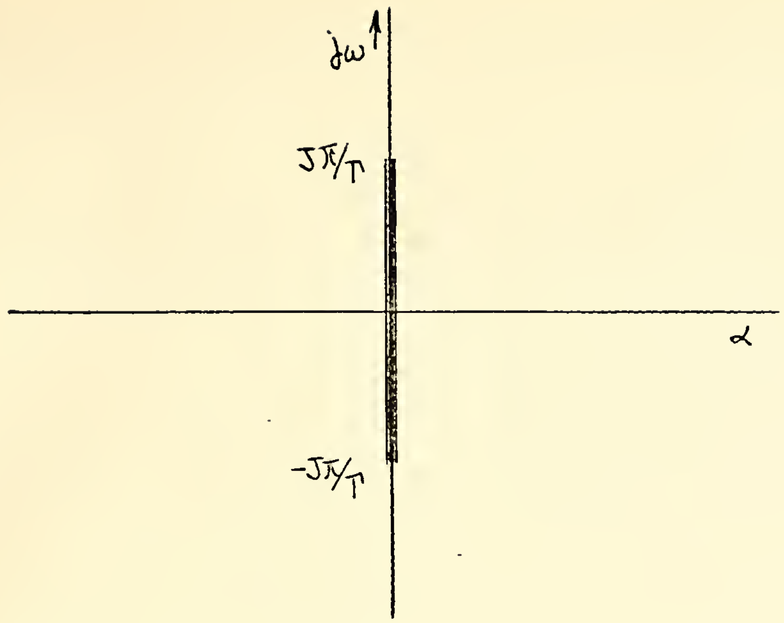



Figure 2: Contour in s-plane for approximate solution

The approximation, $f_A(t)$, of $f(t)$ is given by:

$$f_A(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(s)\, e^{st}\, ds \tag{12}$$

Making $t = nT$

$$f_A(nt) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(s)\, e^{snT}\, ds \tag{13}$$

Solving eq. (6) for s, one obtains:

$$s = 1/T \ln z \tag{14}$$

Making the substitution of (14) into (13) the following equation is obtained

$$f_A(nT) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(1/T \ln z)\, e^{nT(1/T)\ln z}\, d(1/T \ln z) \tag{15}$$

$$f_A(nT) = \frac{1}{2\pi j} \oint \frac{1}{T} F(1/T \ln z)\, z^{n-1}\, dz \tag{16}$$

where the contour, again, is the unit circle in the z-plane.

If eq. (16) is compared with eq. (8), it is seen that it has the form of an inverse z-transform except for the constant multiplier 1/T.

On the other hand, as F(1/T ln z) is a transcendental function, it is not expandable directly into a series of powers of $z^{-1}$ by synthetic division. The synthetic division technique can, however, be used if a proper approximation could be found.

If F(s) is expanded in a series of descending powers of s like

$$F(s) = a_1 s^{-1} + a_2 s^{-2} + \cdots \tag{17}$$

a solution of the form

$$f(t) = a_1 + a_2 t + a_3 \frac{t^2}{2!} + \cdots \tag{18}$$

can be obtained, but this solution is not always valid.

The way to follow is to rearrange F $(1/T)/\ln z$ as a rational function of two polynomials in powers of $s^{-1} = T/\ln z$.

For this purpose, the power series for ln z used is:

$$\ln z = 2[u + 1/3 \cdot u^3 + 1/5 \cdot u^5 + \cdots] \tag{19}$$

where:

$$u = \frac{1 - z^{-1}}{1 + z^{-1}}$$

So the following equation is obtained:

$$\frac{1}{s} = \frac{T}{\ln z} = \frac{T/2}{u + 1/3\ u^3 + 1/5\ u^5} \tag{20}$$

By synthetic division the following Laurent series is obtained:

$$\frac{1}{s} = \frac{T}{\ln z} = \frac{T}{2}\left[\ \frac{1}{u} - \frac{1}{3}\,u - \frac{4}{45}\,u^3 - \frac{44}{945}\,u^5 + \cdots\right] \tag{21}$$

To obtain series expansion of $s^{-k}$, bothsides of (21) can be raised to the $k^{th}$ power.

Table I (on page 15) was obtained by using only the principal part and the constant term of the Laurent series. The result is of the form

TABLE I

| $s^{-k}$ | z-Form = $F_k(z^{-1})$ |
|---|---|
| $s^{-1}$ | $\frac{T}{2}\,\frac{1+z^{-1}}{1-z^{-1}}$ |
| $s^{-2}$ | $\frac{T^2}{12}\,\frac{1+10z^{-1}+z^{-2}}{(1-z^{-1})^2}$ |
| $s^{-3}$ | $\frac{T^3}{2}\,\frac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$ |
| $s^{-4}$ | $\frac{T^4}{6}\,\frac{z^{-1}+4z^{-2}+z^{-3}}{(1-z^{-1})^4} - \frac{T^4}{720}$ |
| $s^{-5}$ | $\frac{T^5}{24}\,\frac{z^{-1}+11z^{-2}+11z^{-3}+z^{-4}}{(1-z^{-1})^5}$ |

$$s^{-k} \quad \frac{Nk\ (z^{-1})}{(1 - z^{-1})^k} = F_k(z^{-1}) \tag{22}$$

where $N_k$ is a polynomial in powers of $z^{-1}$. The right hand side of (24) is called the "z-form" for $s^{-k}$.

<u>Example 1</u> – Third Order System (Fig. 3)

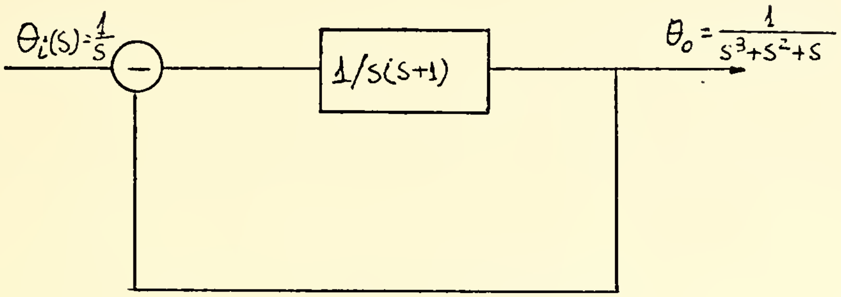



Fig. 3: Typical Feedback System for Example 1

With a step input, as shown, the Laplace transform of the output is:

$$\theta_0(s) = \frac{1}{s^3 + s^2 + s} \tag{23}$$

Following the step by step procedure as given in the Introduction,

$$\theta_0(s) = \frac{s^{-3}}{1 + s^{-1} + s^{-3}} \tag{24}$$

Now, substituting the correspondent forms of $s^{-k}$ from Table I and dividing the result by T:

$$\theta_{OA}{}^*(z) = \frac{6T^2(z^{-1} + z^{-2})}{(12+6T+T^2) - (36+6T-9T^2)z^{-1} + (36-6T-9T^2)z^{-2} - (12-6T+T^2)z^{-3}} \tag{25}$$

The solution is obtained by choosing T, the time interval at which rate our solution is obtained, and proceeding in dividing the denominator into the numerator.

Let us assume T = 1.0, resulting in the following output:

$$\theta_{OA}{}^*(z) = \frac{6z^{-1} + 6z^{-2}}{19 - 33z^{-1} + 21z^{-2} - 7z^{-3}} \tag{26}$$

By long division, the result is:

$$.316z^{-1} + .864z^{-2} + 1.15z^{-3} + 1.16z^{-4} + 1.06z^{-5} + .987z^{-6} + \cdots$$

The accuracy improves if smaller value for T is chosen. Letting T = 0.5 in this same example, substituting on eq. (25):

$$\theta_{OA}{}^*(z) = \frac{1.5z^{-1} + 1.5z^{-2}}{15.25 - 36.75z^{-1} + 30.75z^{-2} - 9.25z^{-3}} \tag{27}$$

carrying out the long division, the result is:

$$0.984z^{-1} + .335z^{-2} + .610z^{-3} + .853z^{-4} + \cdots$$

The points obtained are plotted together with the precise solution in Fig. 4.

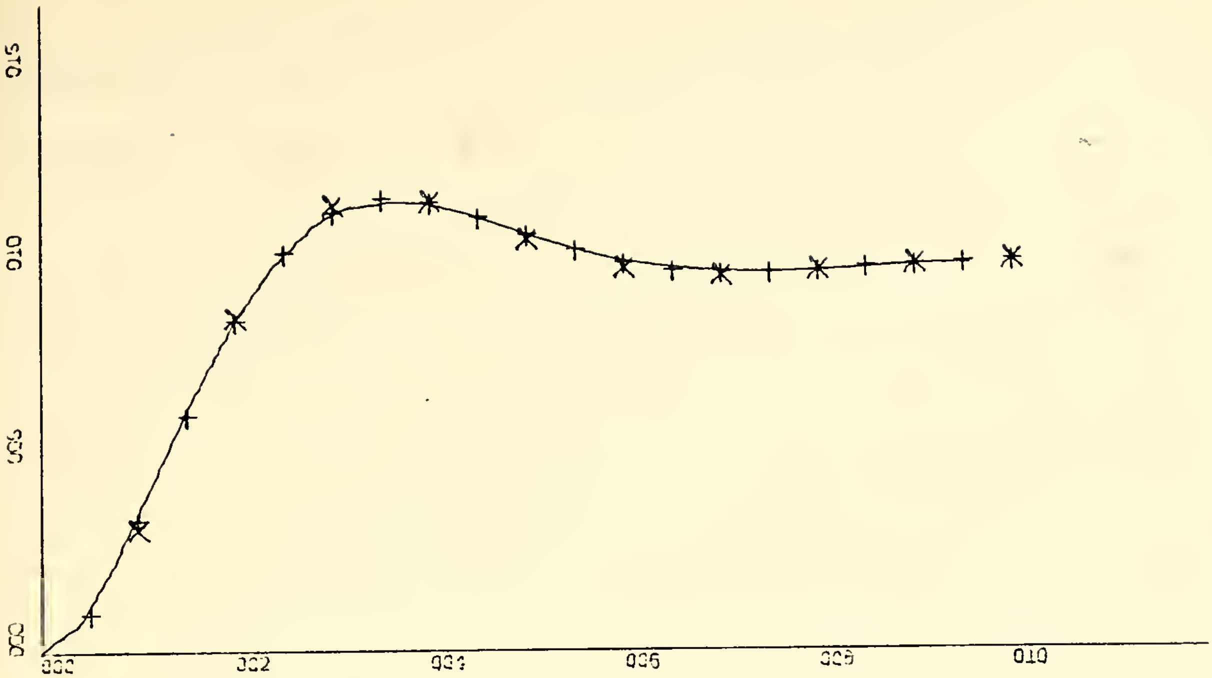


Figure 4

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
x Approximate solution for $\Delta t$ = 1.0
+ Approximate solution for $\Delta t$ = 0.5

<u>Example 2</u> – Solution of Linear Differential Equation

To use the present technique in obtaining the solution of integro-differential equations, one takes the Laplace transform of the equation, initial conditions included as usual, and then proceeds as before. However, after transforming the equation, the following rules can be applied:

1. If the degree of the denominator of the Laplace transform is higher than the degree of the numerator by two or more, proceed directly.

2. If the degree of the numerator is one less, equal to, or greater than the degree of the denominator, divide until the remainder meets the criterion of Rule 1.

Using the procedure stated in the rules above one eliminates the high frequency from F(s), that is implied if the numerator is one degree less than the denominator.

To illustrate this, a simple example is shown:

$$\frac{dY}{dt} + Y = 1 \quad . \tag{28}$$

Assuming for initial condition Y(0) = u, the Laplace transform Y(s) is:

$$Y(s) = \frac{1 + s \cdot u}{s(s+1)} = \frac{s \cdot u + 1}{s^2 + s} \tag{29}$$

Using Rule 2 above, one gets:

$$Y(s) = \frac{u}{s} + \frac{1 - u}{s^2 + s} \tag{30}$$

In this procedure, the step function of amplitude u is added to the result obtained after operating upon $(1 - u)/(s^2 + s)$.

Now, letting u = 0.1, and expressing the second term of eq. (30) in powers of $s^{-1}$,

$$Y_1(s) = \frac{0.9s^{-2}}{1 + s^{-1}} \tag{31}$$

Substituting the z-forms of Table I (on page 15) and dividing by T,

$$Y_{1A}^*(z) = \frac{0.9T(1 + 10z^{-1} + z^{-2})}{(12 + 6T) - 24z^{-1} + (12 - 6T)z^{-2}} \tag{32}$$

Letting T = 0.1 and carrying out the long division one gets:

$$.00714 + .085z^{-1} + .163z^{-2} + .233z^{-3} + .296z^{-4} + \cdots \tag{33}$$

This result is plotted in Fig. 5 after adding the step function of amplitude 0.1.

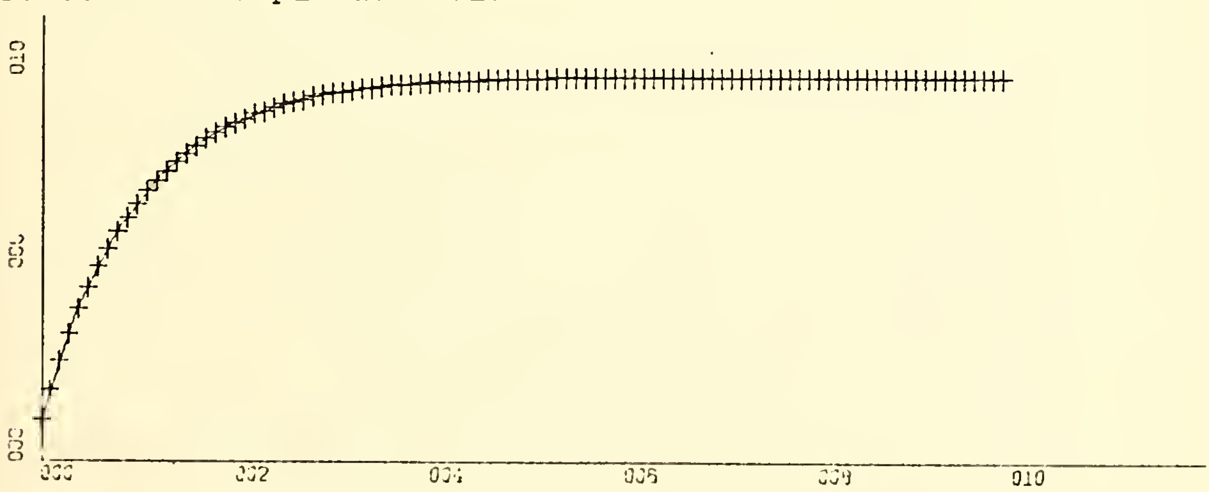



Figure 5

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
\+ Approximate solution

## C. TIME-VARYING SYSTEMS

A linear differential equation with time-varying coefficients may be expressed as

$$\frac{d^n}{dt^n}\left[f_n(t)Y\right] + \frac{d^{n-1}}{dt^{n-1}}\left[f_{n-1}(t)Y\right] + \cdots + f_0(t)Y = f(t) \qquad (34)$$

where $f_n(t)$, $f_{n-1}(t), \ldots, f_0(t)$ are functions of the independent variable t.

To solve this type of equation the following steps may be executed:

1. Obtain the Laplace transform of (34) in the usual manner including initial conditions.

2. Proceed as in the constant coefficient case discussed above. In performing the long division process, however, change the values of $C_n, \ldots, C_0$ at each step in the division process to the values at the corresponding times of the functions

$$\begin{aligned} C_n &= f_n(t) \\ &\vdots \\ C_0 &= f_0(t) \end{aligned} \qquad (35)$$

In general

$$C_i = f_i(t) \qquad (36)$$

## D. NONLINEAR SYSTEMS

A similar manner like used in II.C. may be applied to nonlinear differential equations. The regression equation is obtained by long division process as in the time varying example.

However, the factors $C_n$, $C_{n-1}$, ..., $C_0$ must now represent functions of the dependent variable.

Generally, a nonlinear differential equation can be represented as:

$$\frac{d^n}{dt^n}\left[f_n(Y,\frac{dY}{dt},\ldots)Y\right] + \frac{d^{n-1}}{dt^n}\left[f_{n-1}(Y,\frac{dY}{dt},\ldots)Y\right] + \cdots$$

$$+ \cdots + f_o(Y,\frac{dY}{dt},\ldots)Y = f(t) \tag{37}$$

being the rules for the procedure as follow:

1. Obtain the Laplace transform of (37) in the usual manner including initial conditions.

2. Proceed as in the constant coefficient case. In performing the long division process, however, change the value of $C_n \cdots C_o$ at each step in the division process to the most recent values available for

$$\begin{aligned} C_n &= f_n(Y,\frac{dY}{dt},\ldots) \\ C_{n-1} &= f_{n-1}(Y,\frac{dY}{dt},\ldots) \\ &\vdots \\ C_o &= f_o(Y,\frac{dY}{dt},\ldots) \end{aligned} \tag{38}$$

In general

$$C_i = f_i(Y, \frac{dY}{dt}, \ldots) \tag{39}$$

In other words, in the first step, the values of $C_i$ are obtained from initial conditions for $Y, \frac{dY}{dt}, \ldots$ . The result of the first division is a new value of Y to be used in calculating $C_n, C_{n-1}, \ldots, C_o$ for the second step in the division. New values to be used for the derivatives of Y with respect to t are calculated from the new values of the dependent variables. The process is difficult to describe in words, but is simple in concept.

To illustrate the procedure an example is given:

Example 3:

Let $$\frac{dY}{dt} + Y^2 = 1 \tag{40}$$

Carrying out the procedure outlined above, eq.(40) is written

$$\frac{dY}{dt} + C \cdot Y = 1 \tag{41}$$

The Laplace transform, assuming C to be constant and expressing in powers of $s^{-1}$ is

$$Y(s) = \frac{s^{-2}}{1 + Cs^{-1}} \tag{42}$$

Substituting the z-forms of Table I and dividing by T yields:

$$Y_A^*(z) = \frac{T(1 + 10z^{-1} + z^{-2})}{(12 + 6CT) - 24z^{-1} + (12 - 6CT)z^{-2}} \tag{43}$$

Letting T = 0.1,

$$Y_A^*(z) = \frac{0.1 + 1.0z^{-1} + z^{-2}}{(12 + 0.6C) - 24z^{-1} + (12 - 0.6C)z^{-2}}$$

Carrying out the long division procedure yields:

$$.00833 + 0.100z^{-1} + 0.199z^{-2} + 0.295z^{-3} + 0.390z^{-4} + \ldots \tag{44}$$

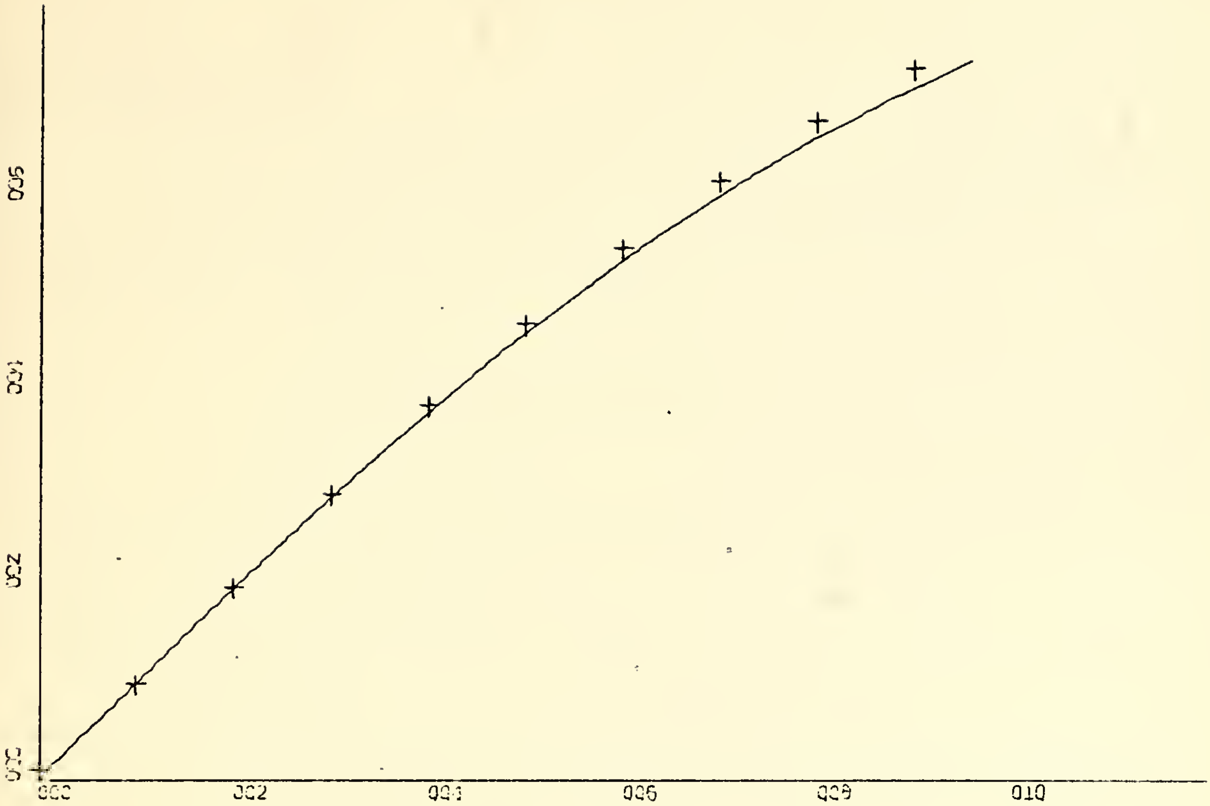


Figure 6

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
+ Approximate solution

## III. CASE STUDIES

### A. INTRODUCTION

In order to test the present technique, a collection of linear differential equations with varying coefficients were studied, whose basic format is:

$$\dot{Y} + P(t)Y = Q(t)$$

where P(t) and Q(t) are time functions. For the purpose of this work, nine equations in all were processed successfully for what P(t) took the values $\sqrt{t}$, t and $t^2$, and Q(t) took the values $1 \cdot u(t)$, t and $t^2$.

The collection of equations, the case nubmers, and the sections in this work, are as follows:

| | | |
|---|---|---|
| $\dot{Y} + \sqrt{t}\,Y = 1$ | Case 1 | |
| $\dot{Y} + tY = 1$ | Case 2 | III B |
| $\dot{Y} + t^2Y = 1$ | Case 3 | |
| $\dot{Y} + \sqrt{t}\,Y = t$ | Case 4 | |
| $\dot{Y} + tY = t$ | Case 5 | III C |
| $\dot{Y} + t^2Y = t$ | Case 6 | |
| $\dot{Y} + \sqrt{t}\,Y = t^2$ | Case 7 | |
| $\dot{Y} + tY = t^2$ | Case 8 | III C |
| $\dot{Y} + t^2Y = t^2$ | Case 9 | |

In Case 10, an example from Ref. 2 was studied, Comments on this problem are given in Case 10 discussion.

In cases 1 to 9 the same type of procedure was followed, applying directly the steps as in II A of this work and experimenting for time intervals form 0.2 sec. to 2.0 sec. in order to show the advantages of this technique with relatively large $\Delta t$'s, its stability together with an algorithm for computer solution and application to hand calculation, i.e., small number of division steps for good results.

B. CASE 1 – $\dot{Y} + \sqrt{t}\, Y = 1$

The equation $\dot{Y} + P(t)Y = Q(t)$ is rewritten for this case as:

$$\frac{dY}{dt} + \sqrt{t}\, Y = 1 \qquad\qquad Y_{(t=0)} = 0$$

following the procedure, as in II A:

$$\frac{dY}{dt} + PY = 1 \qquad\qquad P = \sqrt{t}$$

the Laplace transform is

$$sy + Py = 1/s$$

rearranging and expressing in powers of $s^{-1}$:

$$s^2 y + Psy = 1 \qquad\qquad y(s^2 + Ps) = 1$$

$$y = \frac{1}{s^2 + Ps} = \frac{s^{-2}}{1 + Ps^{-1}}$$

Substituting the z-form values for $s^{-k}$ from Table I (page 15)

$$y_A^*(z) = \frac{\dfrac{T^2}{12}\ \dfrac{1 + 10z^{-1} + z^{-2}}{(1 - z^{-1})^2}}{1 + P\ \dfrac{T}{2}\ \dfrac{1 + z^{-1}}{1 - z^{-1}}}$$

Dividing by T and rearranging:

$$y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}} \qquad (45)$$

where again P is the time varying coefficient of Y in the differential equation and T is the iteration time, or in other words the time difference between results.

The following step consists of dividing the denominator into the numerator and substituting in P, its value for the corresponding time ($t = nT$). As the exponent of z for the first value in the quotient is zero, $P(t) = \sqrt{t}$ at $t = 0$, $P(t) = 0$ in the first division step.

The value for T first chosen was 0.2 sec. .

The algorithm used for computing the quotient (Program 1, page 30) as the solution, for values of T from 0.2 sec. up to 2 sec. with the "precise" solution in continuous line and with the approximate solution as crosses follow; the "precise" solution was obtained by computer, using the subroutine called INTEG1.

Scales in the graphs (Figs. 7 to 20) are the same in each case.

In the numerical output tables, the time is referred to as T, the approximate solution is referred to as Q(N) and the "precise" solution as PR. TFIN is the final time.

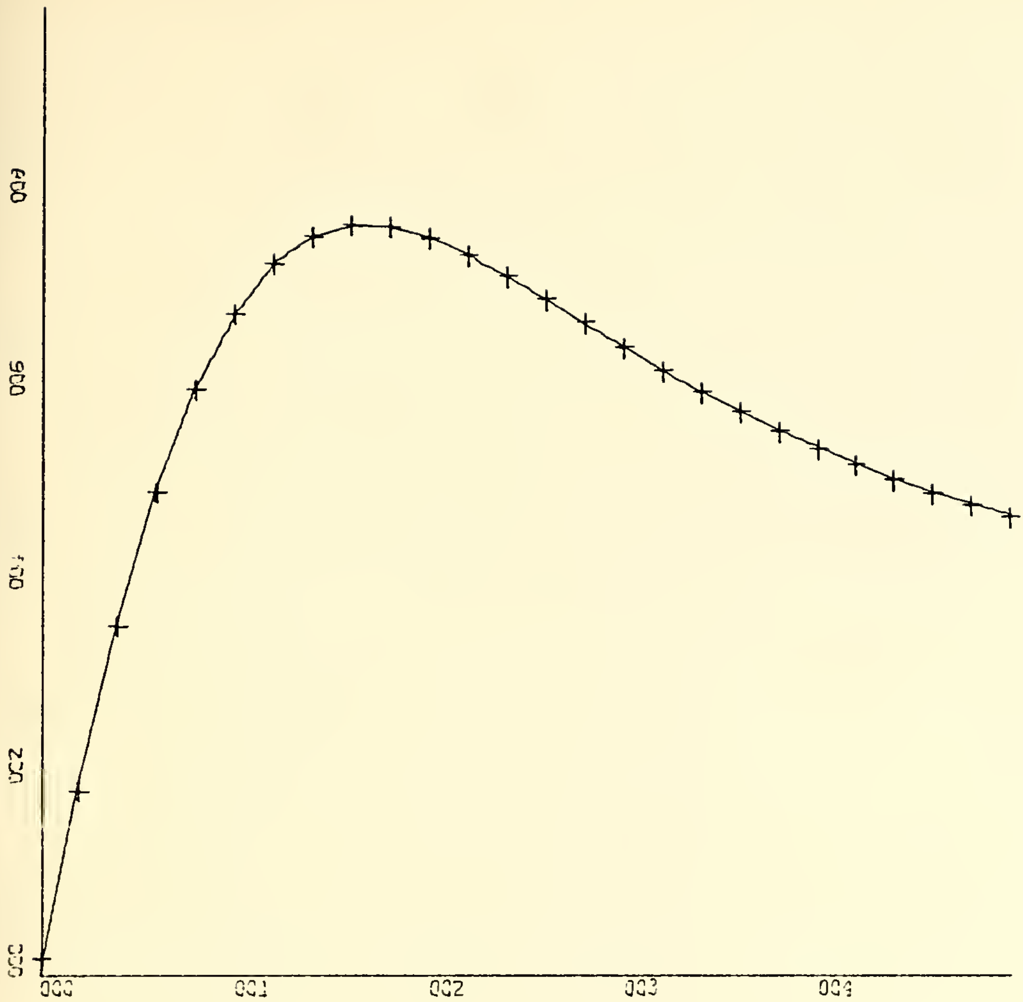
It can be seen that the approximated solution even for large time iterations is very close to the precise solution.

FOR DELTA T = 0.20 TFIN = 5.2

| T | Q(N) | PR |
|---|---|---|
| 0.0 | 0.01667 | 0.0 |
| 0.2 | 0.19144 | 0.19300 |
| 0.4 | 0.36010 | 0.36197 |
| 0.6 | 0.49870 | 0.50039 |
| 0.8 | 0.60588 | 0.60716 |
| 1.0 | 0.68335 | 0.68414 |
| 1.2 | 0.73455 | 0.73486 |
| 1.4 | 0.76372 | 0.76363 |
| 1.6 | 0.77528 | 0.77491 |
| 1.8 | 0.77345 | 0.77289 |
| 2.0 | 0.76192 | 0.76128 |
| 2.2 | 0.74384 | 0.74318 |
| 2.4 | 0.72170 | 0.72109 |
| 2.6 | 0.69743 | 0.69689 |
| 2.8 | 0.67245 | 0.67200 |
| 3.0 | 0.64773 | 0.64738 |
| 3.2 | 0.62392 | 0.62366 |
| 3.4 | 0.60141 | 0.60122 |
| 3.6 | 0.58039 | 0.58027 |
| 3.8 | 0.56092 | 0.56085 |
| 4.0 | 0.54297 | 0.54294 |
| 4.2 | 0.52648 | 0.52647 |
| 4.4 | 0.51132 | 0.51133 |
| 4.6 | 0.49738 | 0.49741 |
| 4.8 | 0.48454 | 0.48458 |
| 5.0 | 0.47268 | 0.47272 |

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=STEP , P(T)=SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = SQRT(T)
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C   Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,5) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,5) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    7 FORMAT ('   T=',7X,' Q(N)=')
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

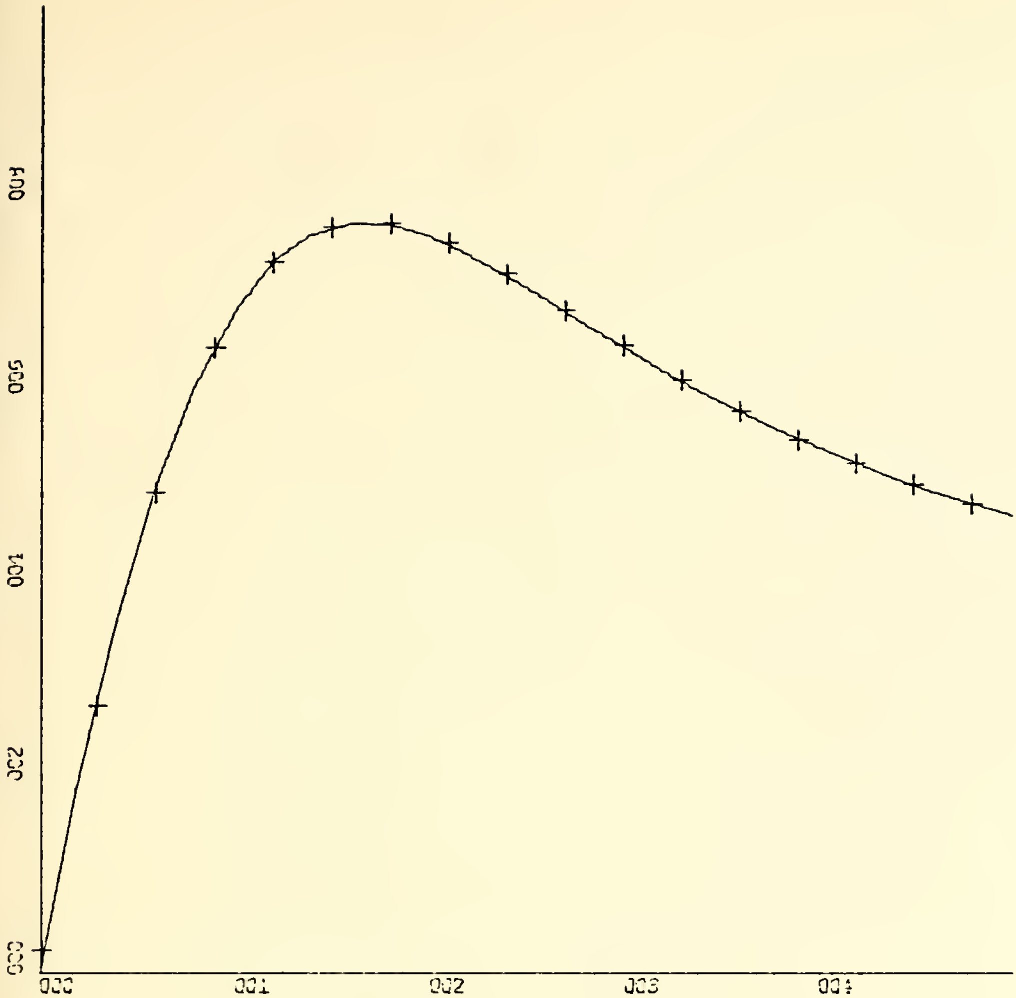
PROGRAM 1



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

FIGURE 7

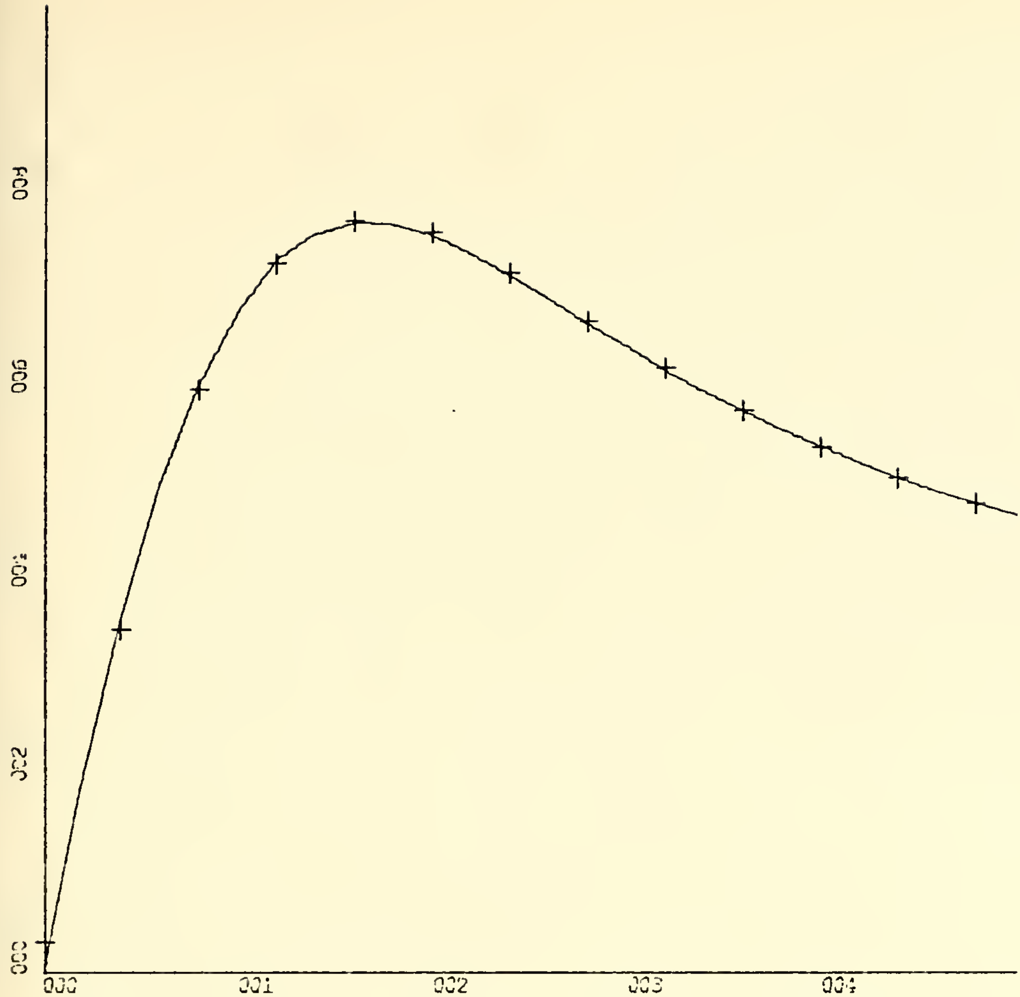


x-scale = 1.0 units/inch

FOR DELTA T = 0.3 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.02500 |
| 0.3 | 0.27722 |
| 0.6 | 0.49673 |
| 0.9 | 0.64695 |
| 1.2 | 0.73424 |
| 1.5 | 0.77180 |
| 1.8 | 0.77421 |
| 2.1 | 0.75441 |
| 2.4 | 0.72252 |
| 2.7 | 0.68563 |
| 3.0 | 0.64822 |
| 3.3 | 0.61282 |
| 3.6 | 0.58059 |
| 3.9 | 0.55187 |
| 4.2 | 0.52653 |
| 4.5 | 0.50422 |
| 4.8 | 0.48454 |

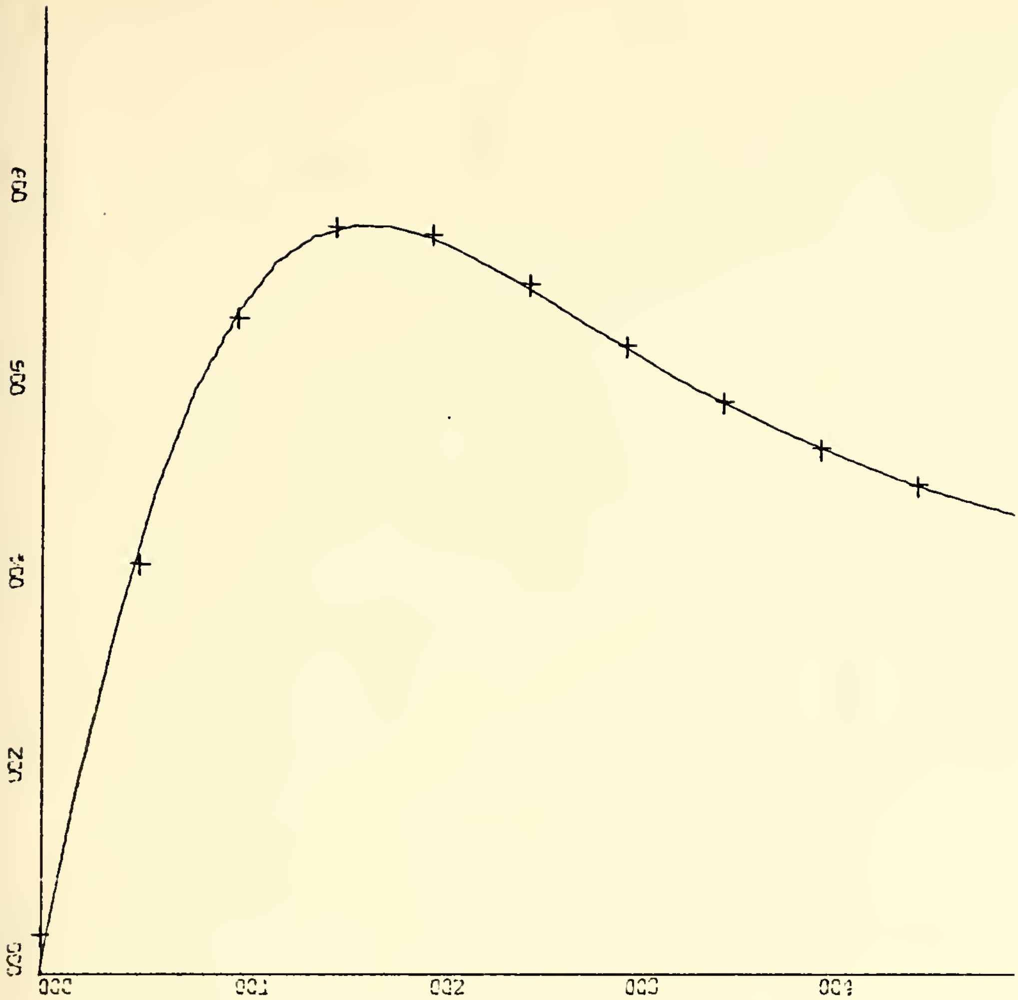
FIGURE 8



FOR DELTA T = 0.40      TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.03333 |
| 0.4 | 0.35508 |
| 0.8 | 0.60241 |
| 1.2 | 0.73386 |
| 1.6 | 0.77661 |
| 2.0 | 0.76404 |
| 2.4 | 0.72370 |
| 2.8 | 0.67393 |
| 3.2 | 0.62484 |
| 3.6 | 0.58087 |
| 4.0 | 0.54317 |
| 4.4 | 0.51137 |
| 4.8 | 0.48453 |

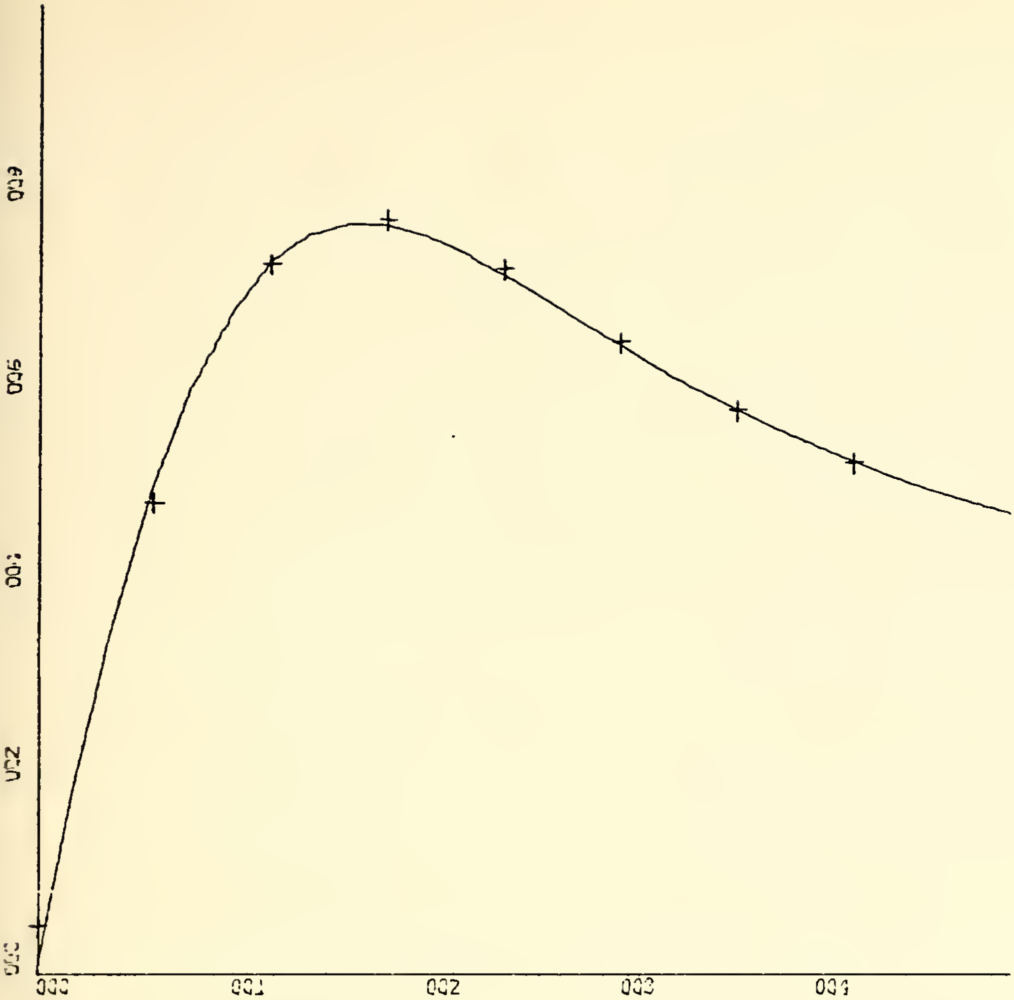
FIGURE 9



FOR DELTA T = 0.5 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.4167 |
| 0.5 | 0.42489 |
| 1.0 | 0.67982 |
| 1.5 | 0.77314 |
| 2.0 | 0.76570 |
| 2.5 | 0.71310 |
| 3.0 | 0.64984 |
| 3.5 | 0.59170 |
| 4.0 | 0.54331 |
| 4.5 | 0.50424 |

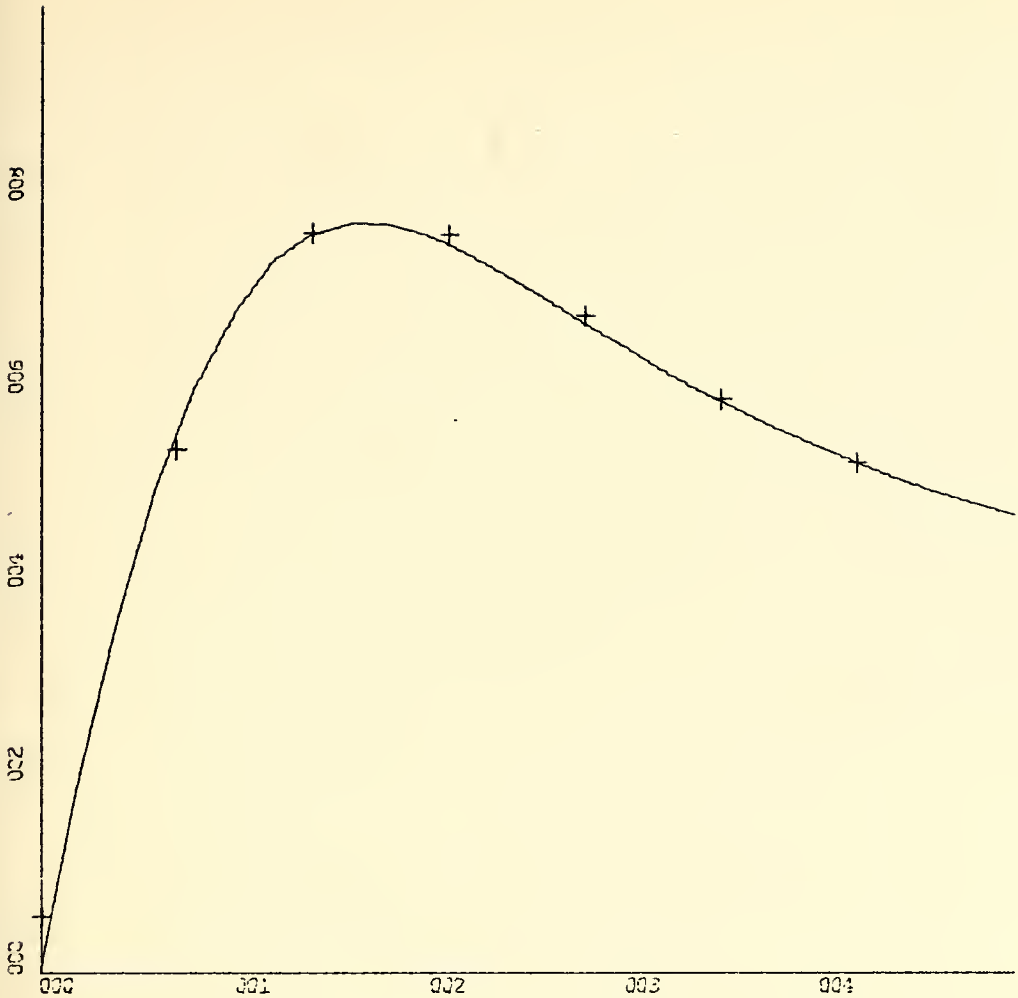
FIGURE 10



FOR DELTA T = 0.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.05000 |
| 0.6 | 0.48686 |
| 1.2 | 0.73288 |
| 1.8 | 0.77864 |
| 2.4 | 0.72725 |
| 3.0 | 0.65099 |
| 3.6 | 0.58165 |
| 4.2 | 0.52673 |

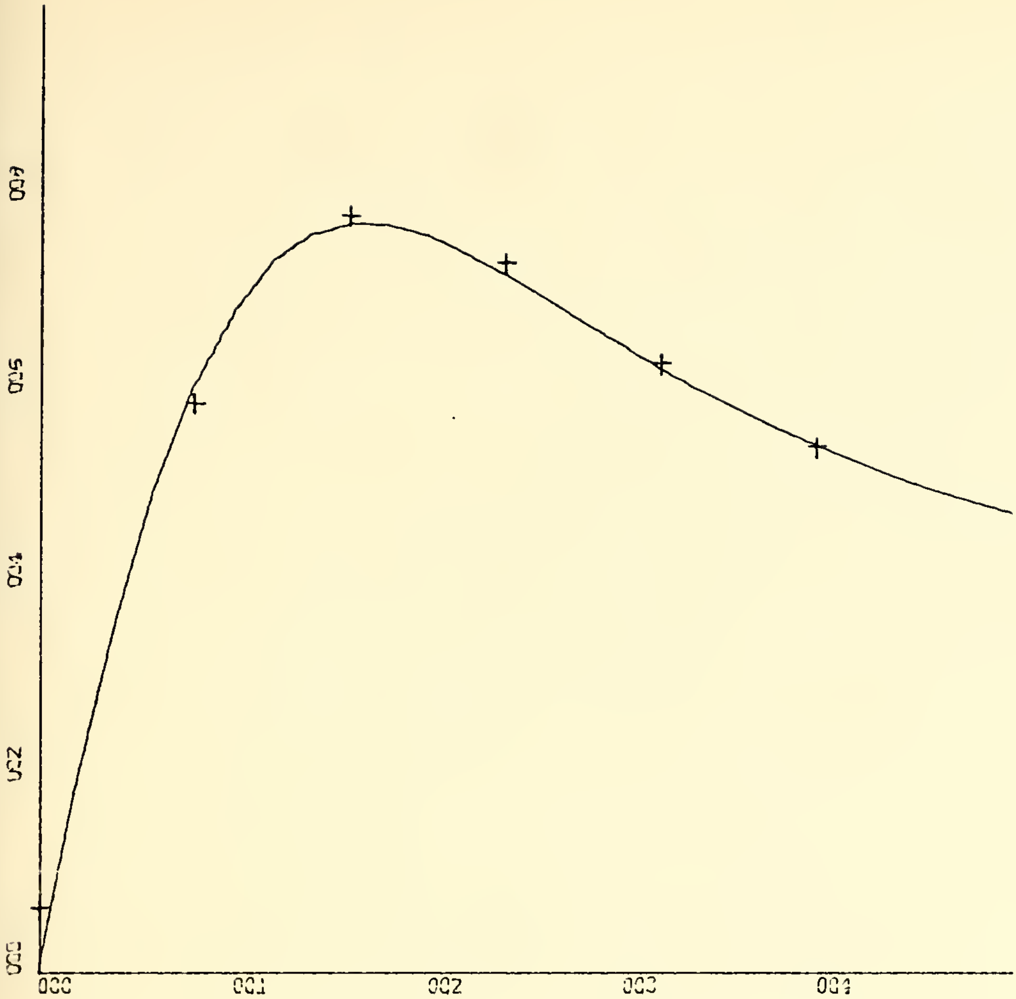
FIGURE 11



FOR DELTA T = 0.7 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.5833 |
| 0.7 | 0.54145 |
| 1.4 | 0.76577 |
| 2.1 | 0.76211 |
| 2.8 | 0.67831 |
| 4.2 | 0.52680 |

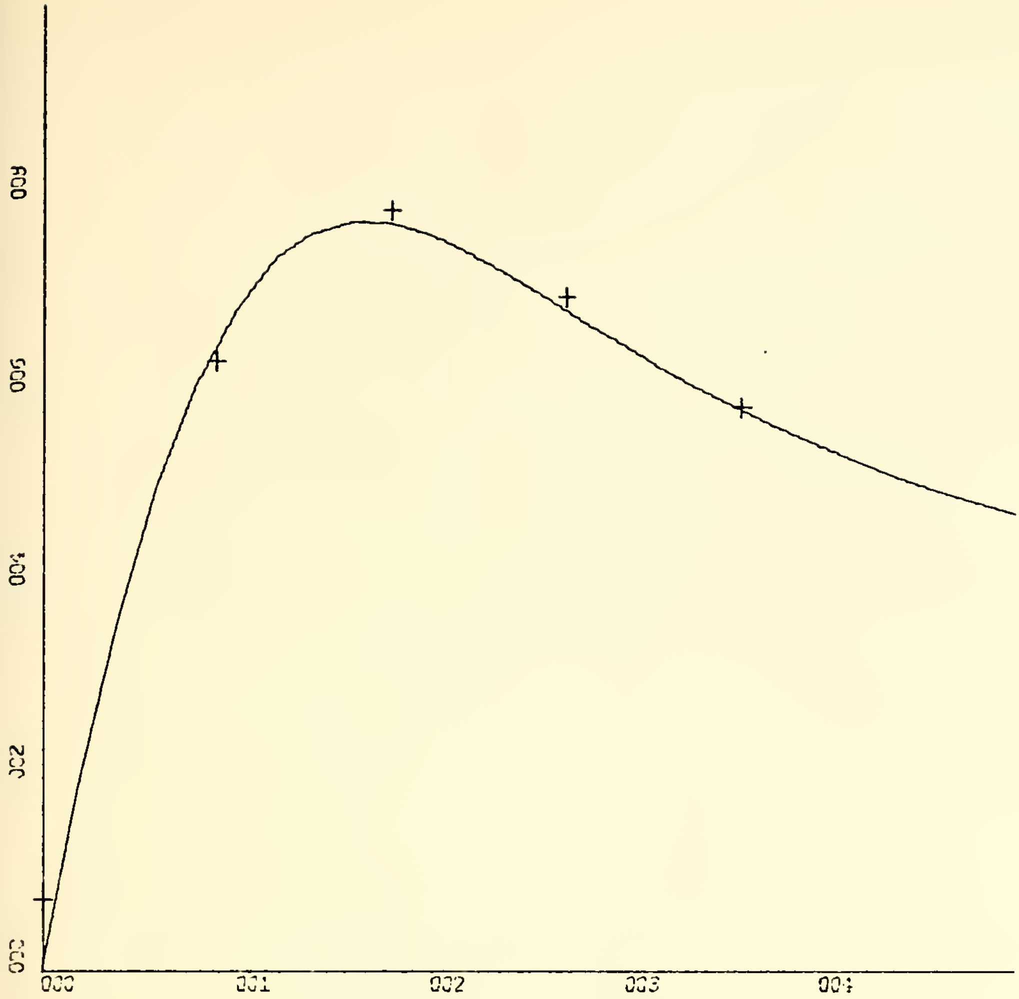
FIGURE 12



FOR DELTA T = 0.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.06667 |
| 0.8 | 0.58920 |
| 1.6 | 0.78249 |
| 2.4 | 0.73260 |
| 3.2 | 0.62874 |
| 4.0 | 0.54381 |

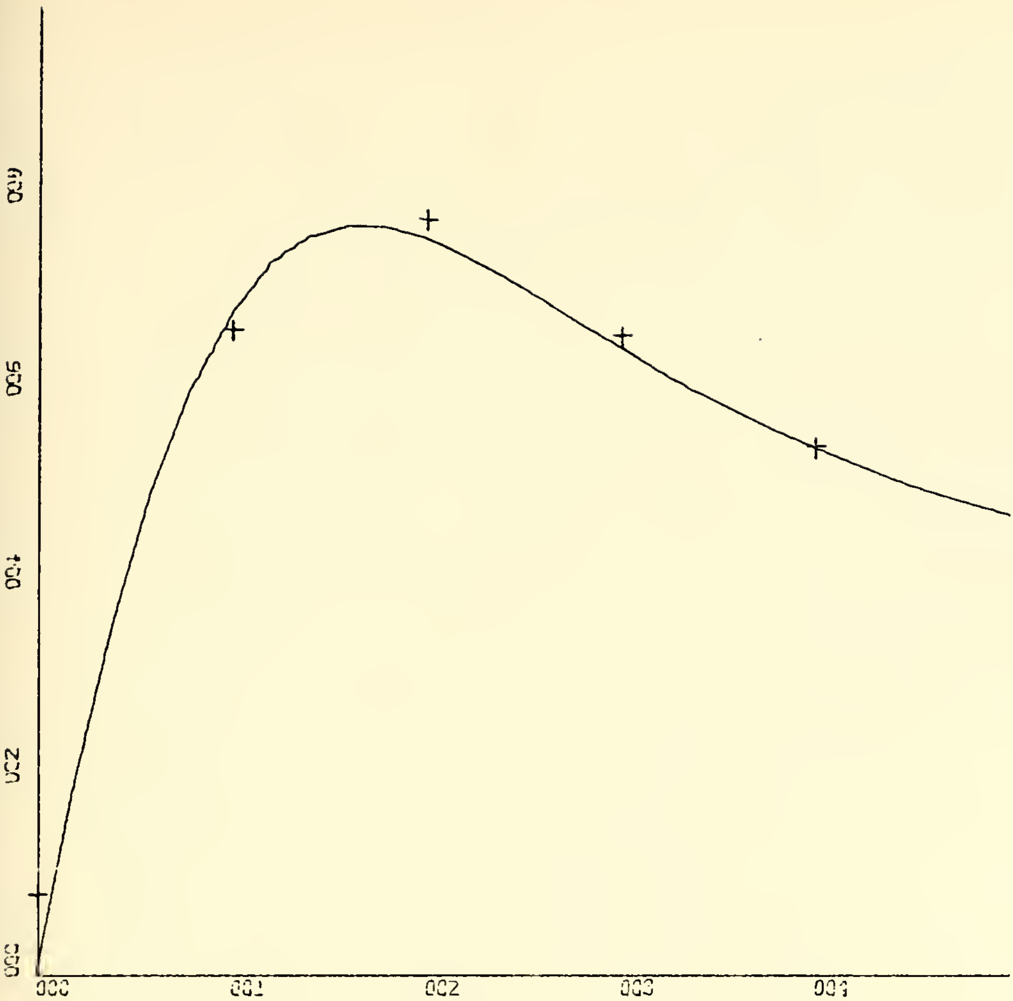
FIGURE 13



FOR DELTA T = 0.9 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.07500 |
| 0.9 | 0.63073 |
| 1.8 | 0.78658 |
| 2.7 | 0.69661 |
| 3.6 | 0.58340 |

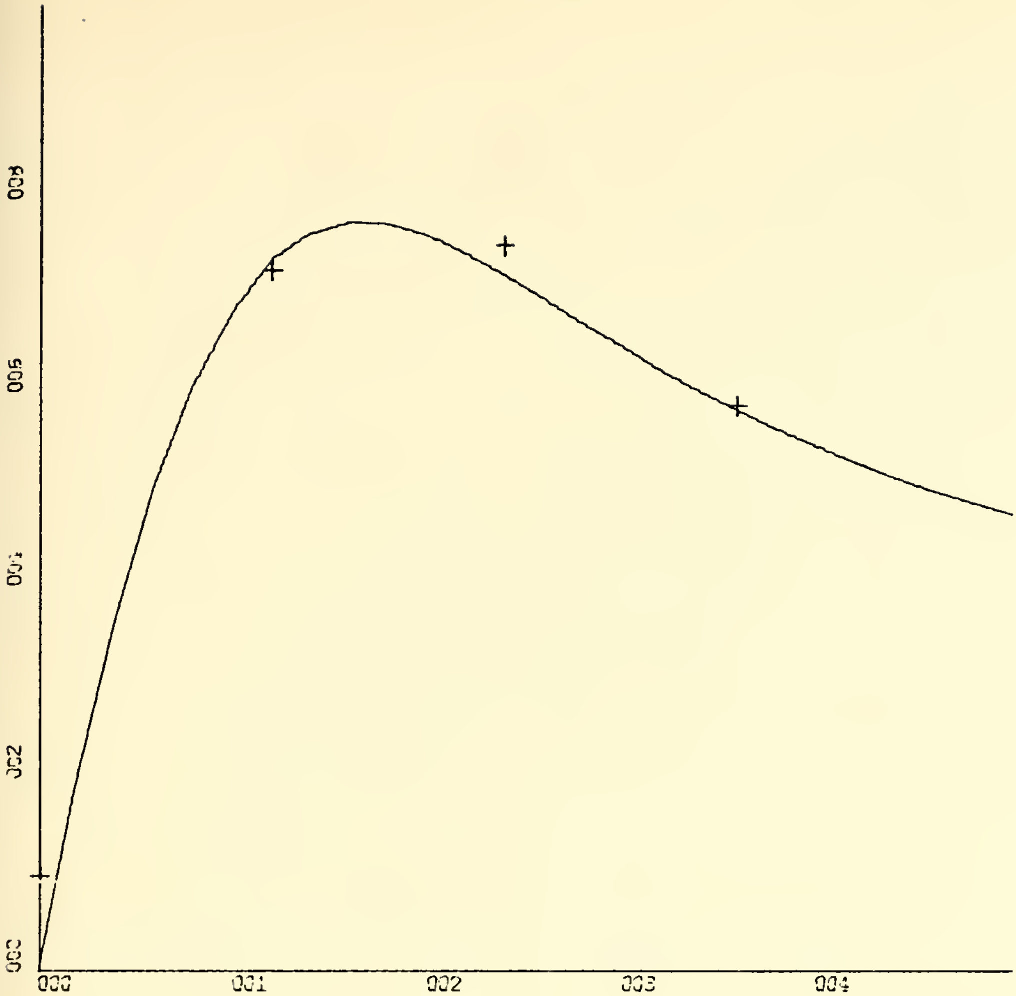
FIGURE 14



FOR DELTA T = 1.0     TFIN + 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.08333 |
| 1.0 | 0.66667 |
| 2.0 | 0.78105 |
| 3.0 | 0.65849 |
| 4.0 | 0.54411 |

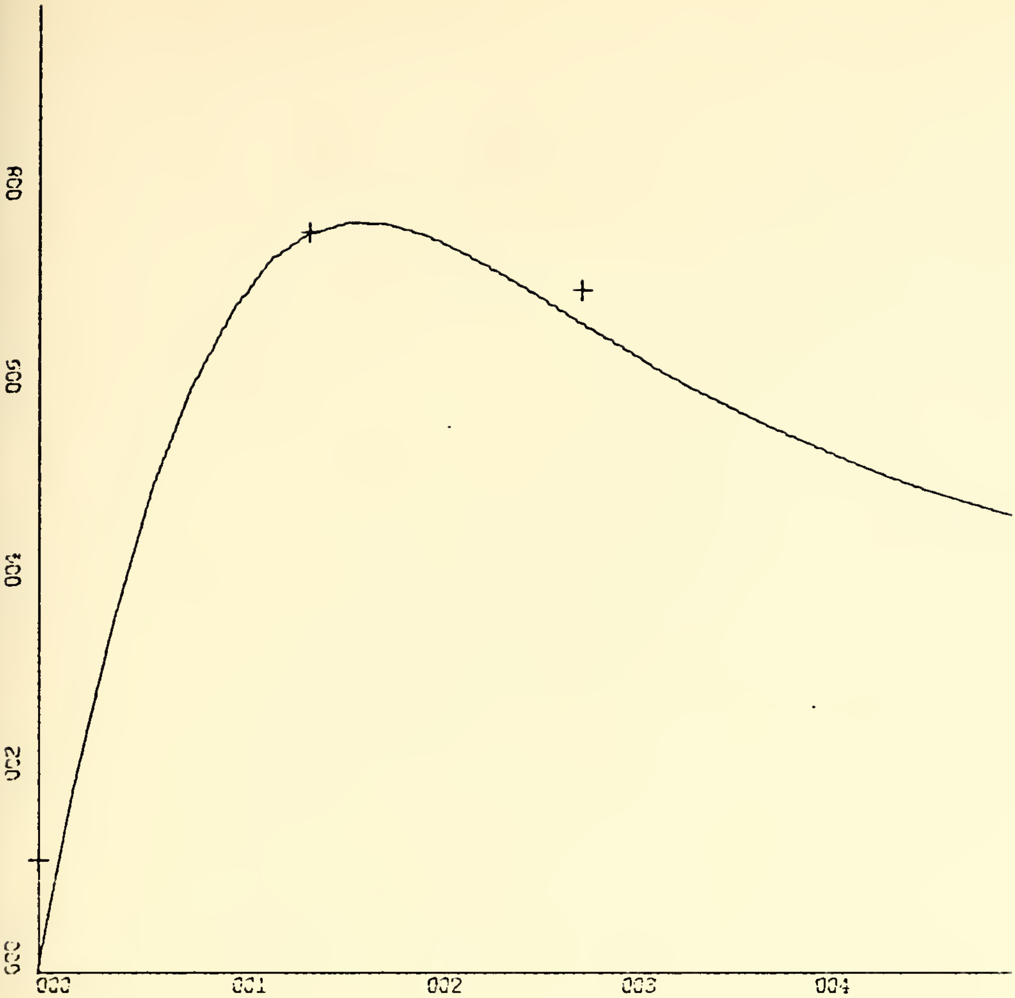
FIGURE 15



FOR DELTA T = 1.2 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.10000 |
| 1.2 | 0.72408 |
| 2.4 | 0.75053 |
| 3.6 | 0.58590 |

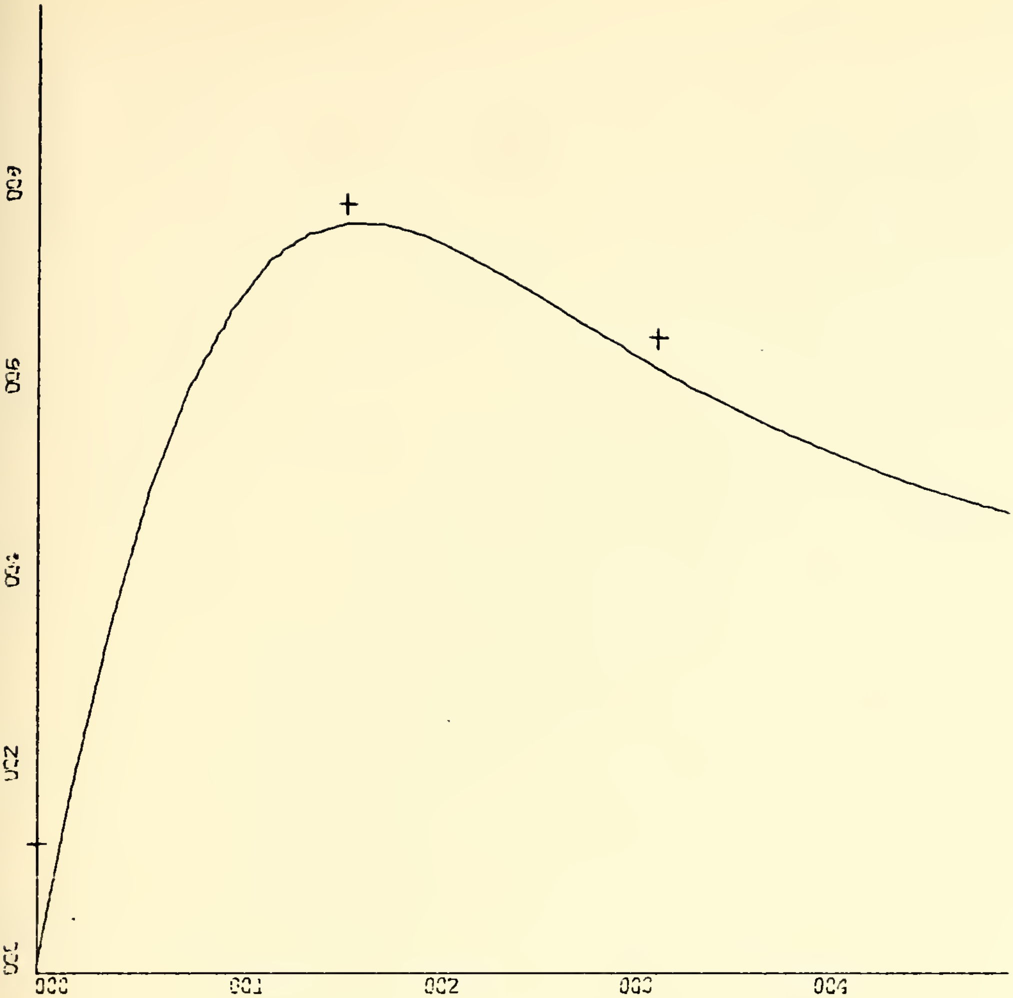
FIGURE 16



FOR DELTA T = 1.4     TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.11667 |
| 1.4 | 0.76576 |
| 2.8 | 0.70534 |

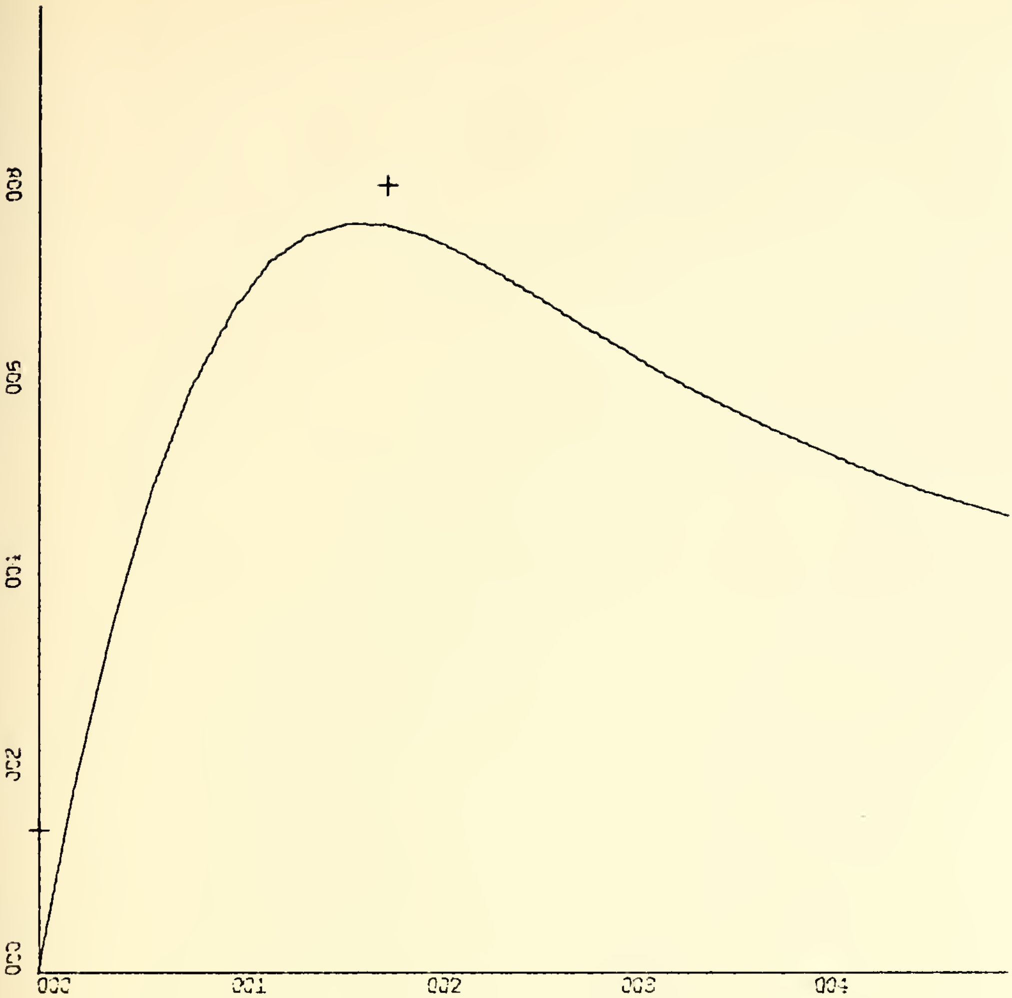
FIGURE 17



FOR DELTA T = 1.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.13333 |
| 1.6 | 0.79526 |
| 3.2 | 0.65424 |

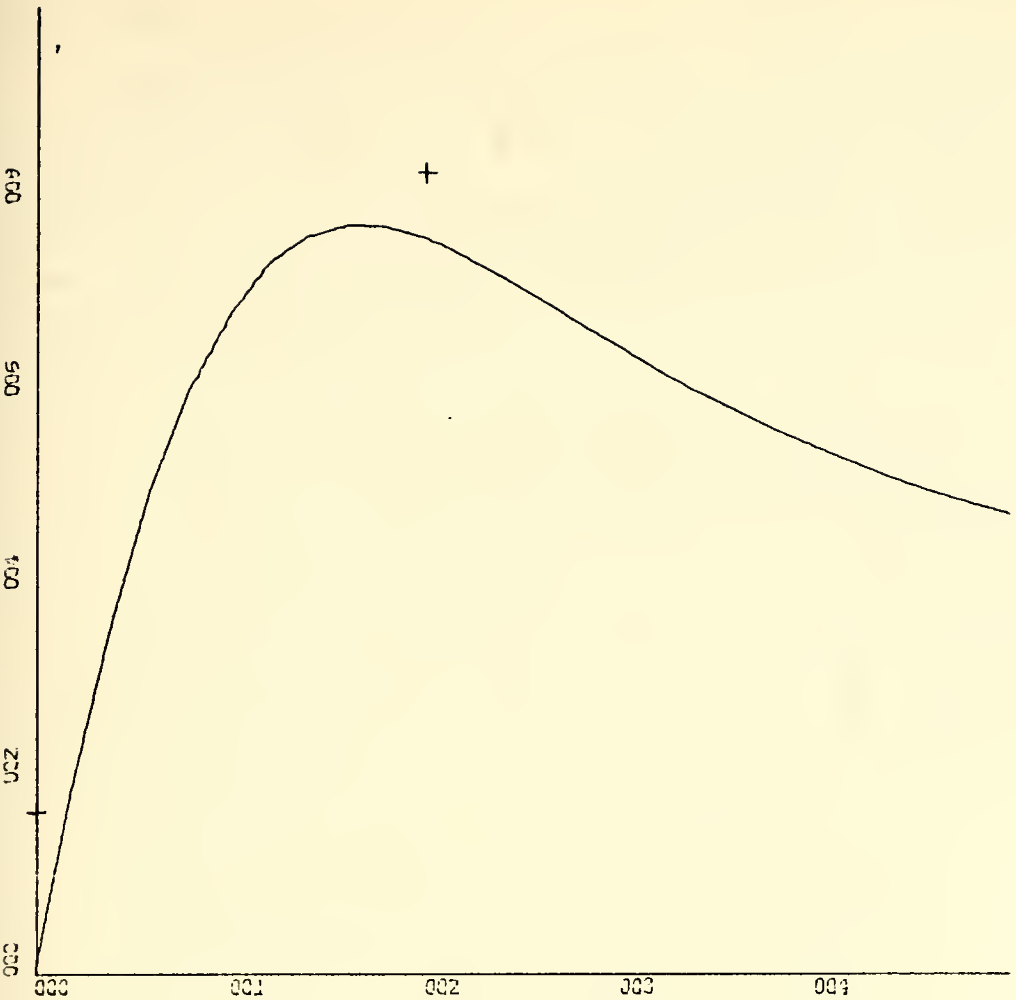
FIGURE 18



FOR DELTA T = 1.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.15000 |
| 1.8 | 0.81541 |

FIGURE 19



DELTA T = 2.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.16667 |
| 2.0 | 0.82843 |

FIGURE 20

1. Case 2 – $\dot{Y} + tY = 1$

The calculations are the same as in Case 1 with the only difference that P(t) is now equal to t. So, when dividing as in Equation (45), repeated here for convenience, the denominator into the numerator the substitution for each step is as P = t.

$$Y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}}$$

In the same manner, numerical and graphical output for T from 0.2 sec. to 2.0 sec. (Figs. 21 to 34) follow. Program 2 (page 47 ) shows a computer algorithm for solution of this problem.

FOR DELTA T = 0.2 TFIN = 5.2

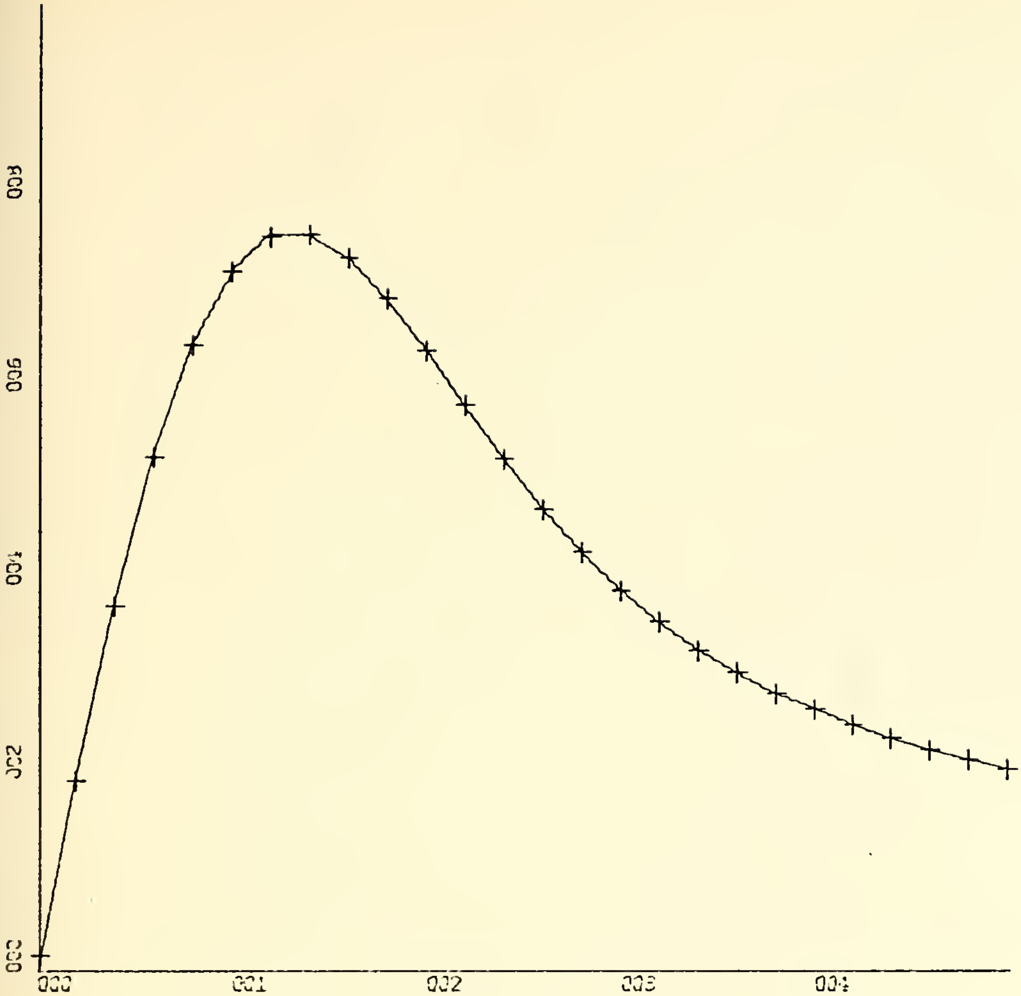
| T | Q(N) | PR |
|---|---|---|
| 0.0 | 0.01667 | 0.0 |
| 0.2 | 0.19608 | 0.19735 |
| 0.4 | 0.37707 | 0.37933 |
| 0.6 | 0.53018 | 0.53293 |
| 0.8 | 0.64664 | 0.64931 |
| 1.0 | 0.72264 | 0.72478 |
| 1.2 | 0.75927 | 0.76058 |
| 1.4 | 0.76154 | 0.76198 |
| 1.6 | 0.73700 | 0.73670 |
| 1.8 | 0.69413 | 0.69335 |
| 2.0 | 0.64099 | 0.64000 |
| 2.2 | 0.58425 | 0.58329 |
| 2.4 | 0.52880 | 0.52801 |
| 2.6 | 0.47769 | 0.47713 |
| 2.8 | 0.43241 | 0.43208 |
| 3.0 | 0.39333 | 0.39319 |
| 3.2 | 0.36010 | 0.36008 |
| 3.4 | 0.33199 | 0.33204 |
| 3.6 | 0.30817 | 0.30825 |
| 3.8 | 0.28784 | 0.28793 |

FOR DELTA T = 0.2 TFIN = 5.2
(Continued)

| T | Q(N) | PR |
|---|---|---|
| 4.0 | 0.27033 | 0.27041 |
| 4.2 | 0.25507 | 0.25514 |
| 4.4 | 0.24162 | 0.24168 |
| 4.6 | 0.22966 | 0.22971 |
| 4.8 | 0.21893 | 0.21897 |
| 5.0 | 0.29023 | 0.20926 |

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=STEP , P(T)=T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = T
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C   Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,7) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,7) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('   XX=',7X,'    PR=')
      END
```

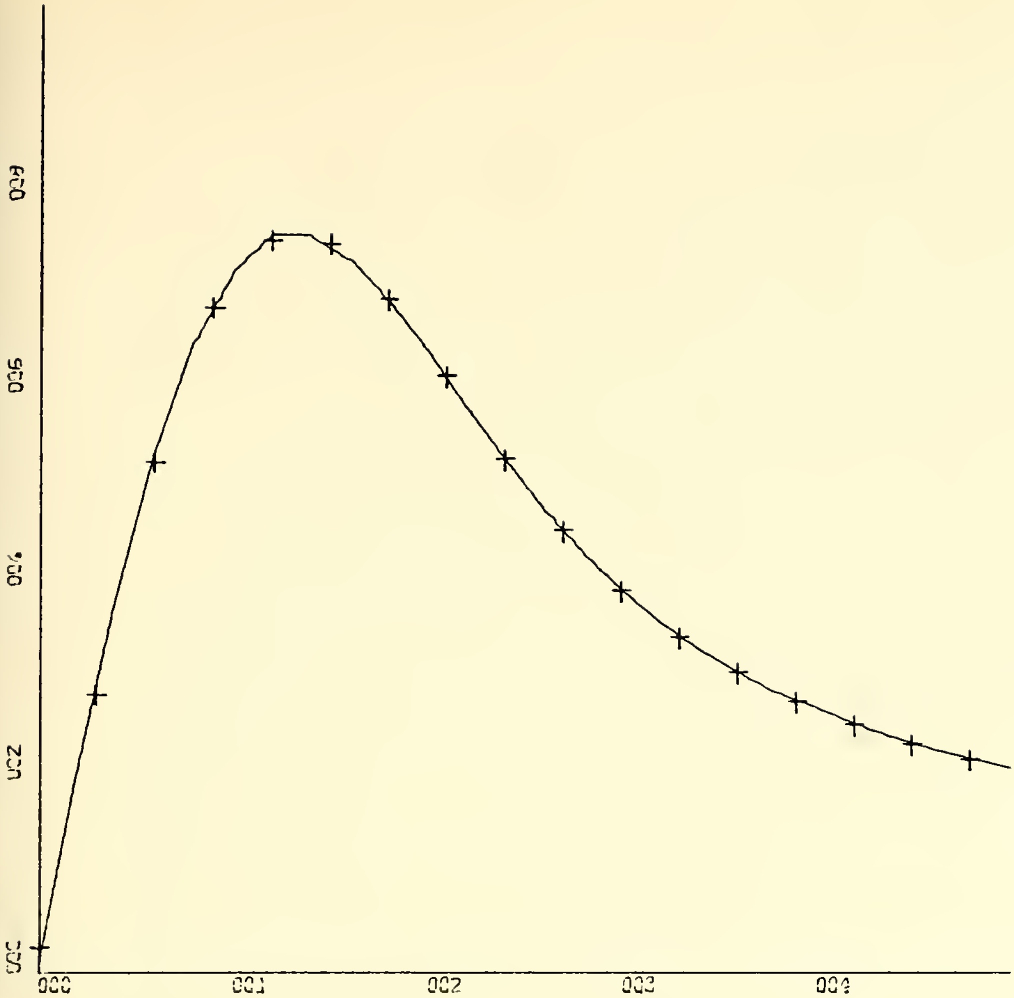
PROGRAM 2



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

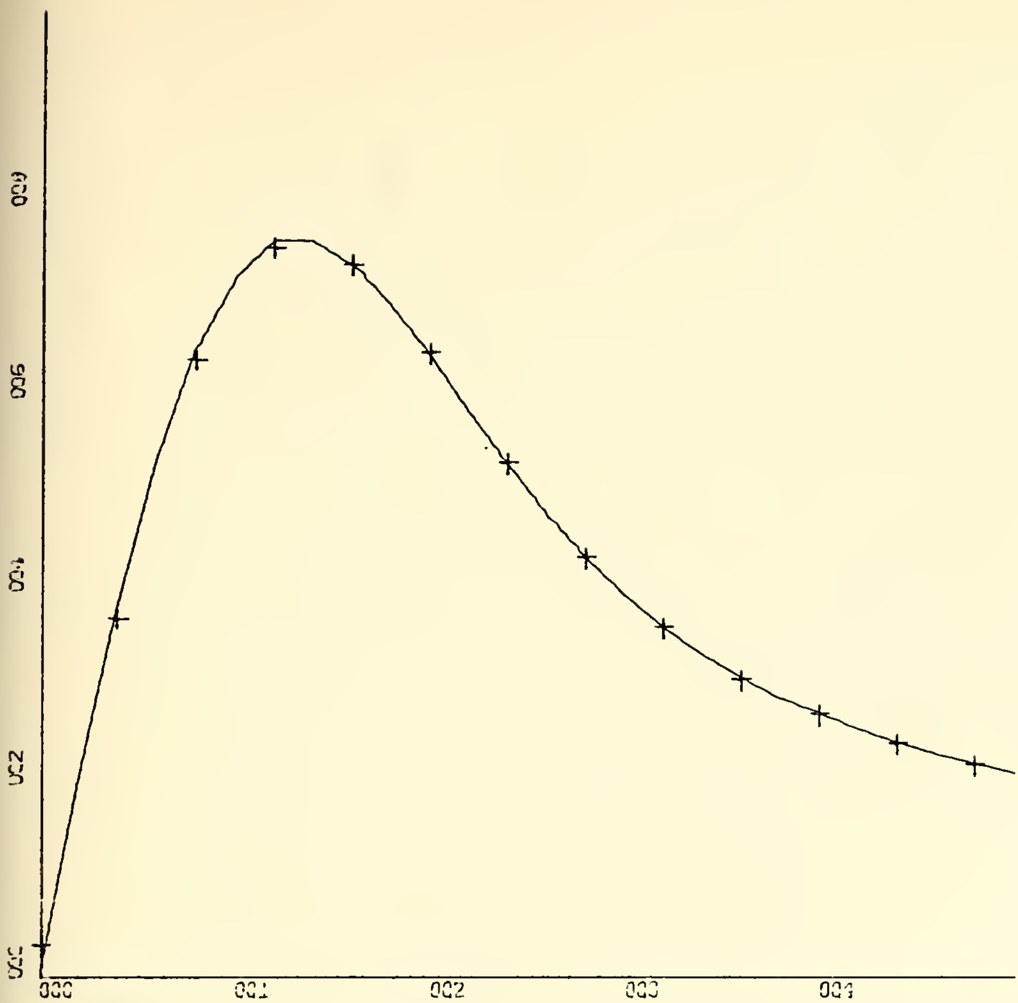
FIGURE 21



FOR DELTA T = 0.3 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.02500 |
| 0.3 | 0.28708 |
| 0.6 | 0.52675 |
| 0.9 | 0.68665 |
| 1.2 | 0.75759 |
| 1.5 | 0.75202 |
| 1.8 | 0.69513 |
| 2.1 | 0.61402 |
| 2.4 | 0.52986 |
| 2.7 | 0.45488 |
| 3.0 | 0.39355 |
| 3.3 | 0.34545 |
| 3.6 | 0.30809 |
| 3.9 | 0.27869 |
| 4.2 | 0.25500 |
| 4.5 | 0.23543 |
| 4.8 | 0.21891 |

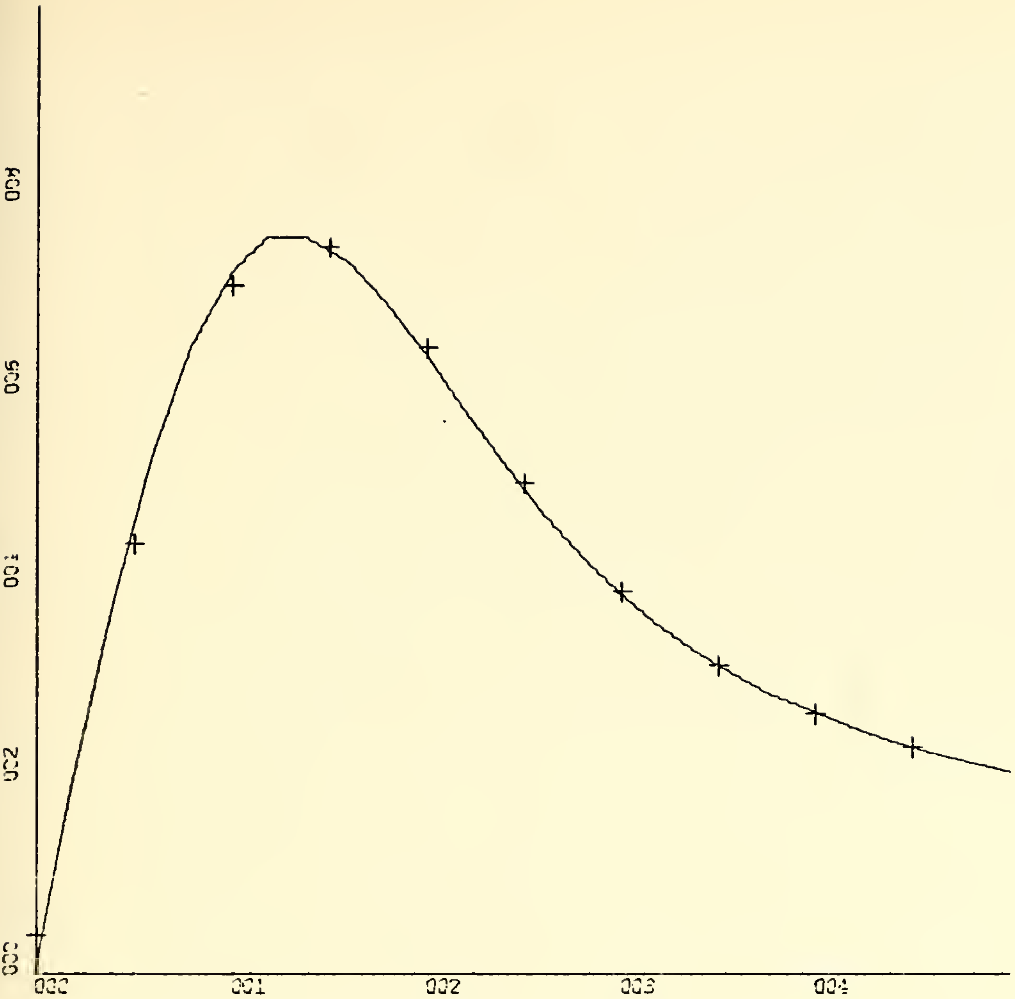
FIGURE 22



FOR DELTA T = 0.40 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.03333 |
| 0.4 | 0.37037 |
| 0.8 | 0.63857 |
| 1.2 | 0.75516 |
| 1.6 | 0.73782 |
| 2.0 | 0.64408 |
| 2.4 | 0.53138 |
| 2.8 | 0.43354 |
| 3.2 | 0.36022 |
| 3.6 | 0.30795 |
| 4.0 | 0.27013 |
| 4.4 | 0.24150 |
| 4.8 | 0.21887 |

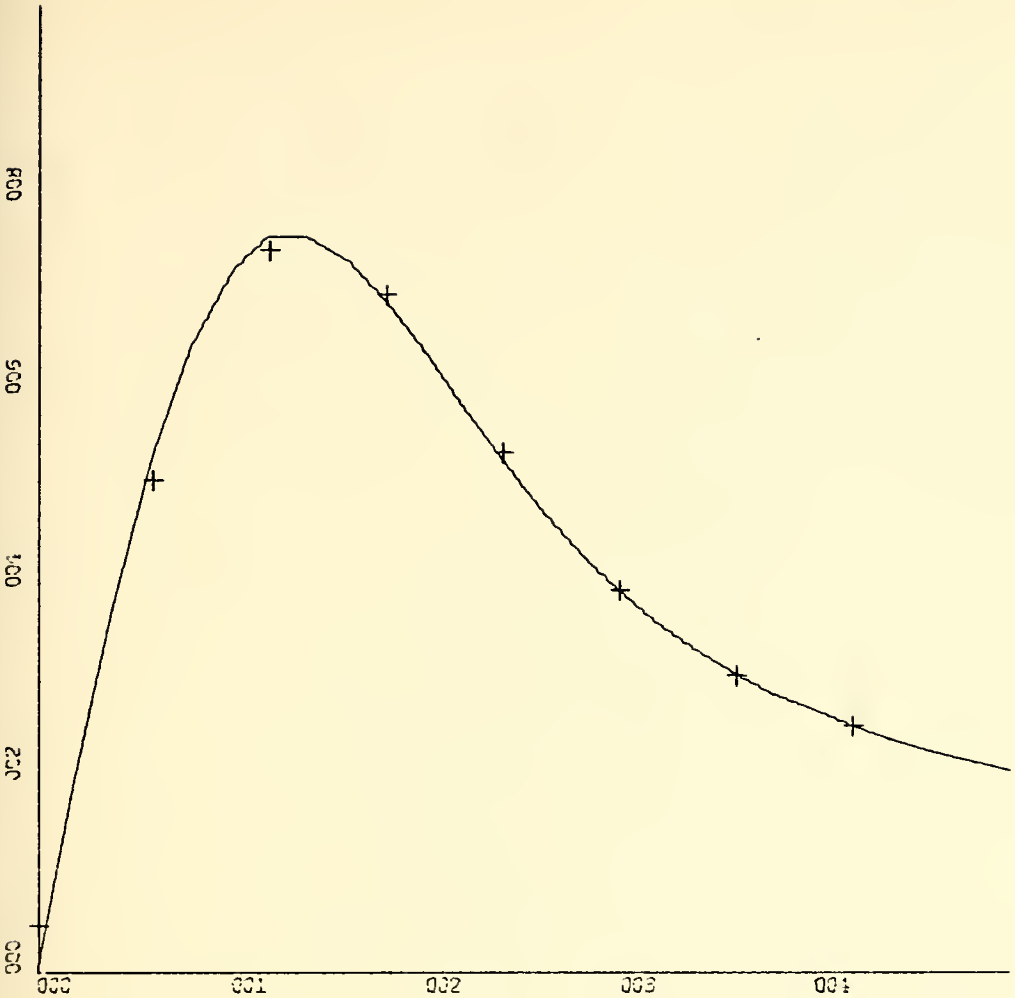
FIGURE 23



FOR DELTA T = 0.5 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.04167 |
| 0.5 | 0.44444 |
| 1.0 | 0.71111 |
| 1.5 | 0.75151 |
| 2.0 | 0.64646 |
| 2.5 | 0.50660 |
| 3.0 | 0.39427 |
| 3.5 | 0.31923 |
| 4.0 | 0.26995 |
| 4.5 | 0.23529 |

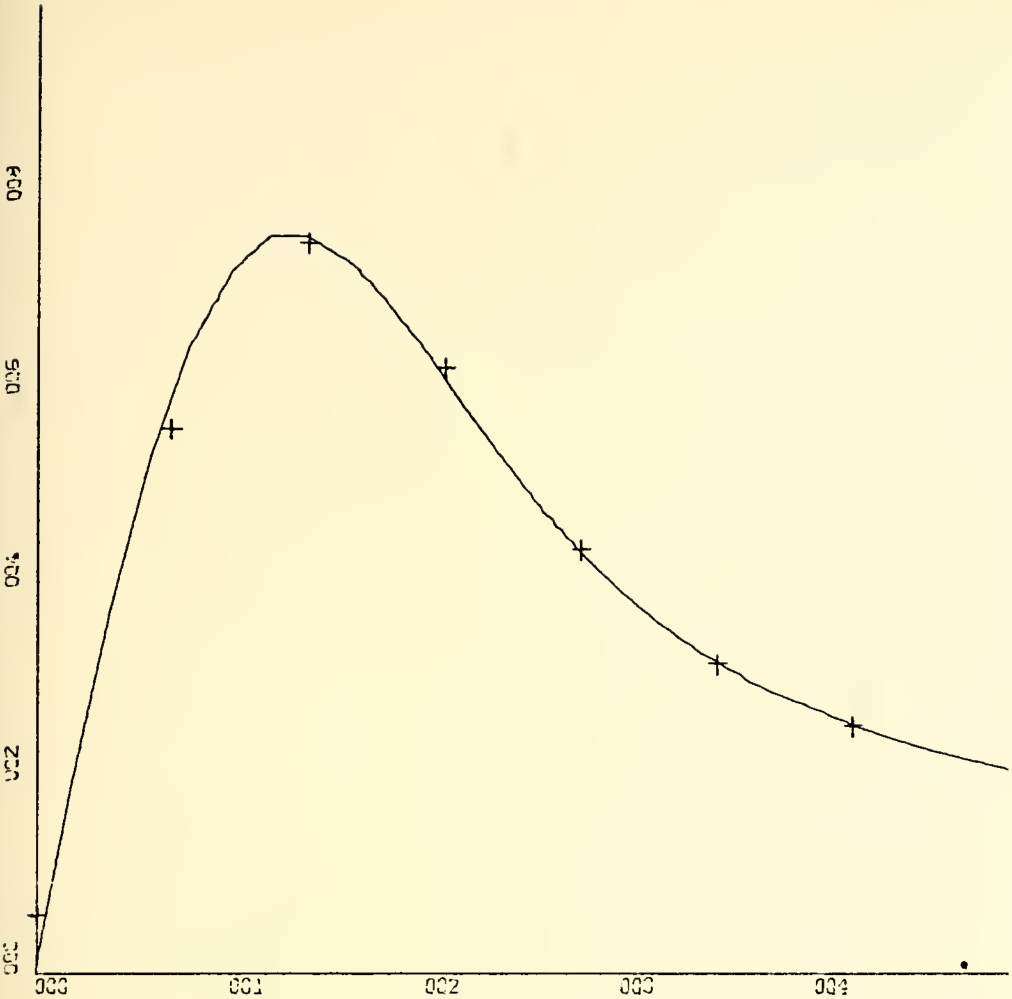
FIGURE 24



FOR DELTA T = 0.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.5000 |
| 0.6 | 0.50847 |
| 1.2 | 0.74776 |
| 1.8 | 0.70037 |
| 2.4 | 0.53615 |
| 3.0 | 0.39480 |
| 3.6 | 0.30744 |
| 4.2 | 0.25460 |

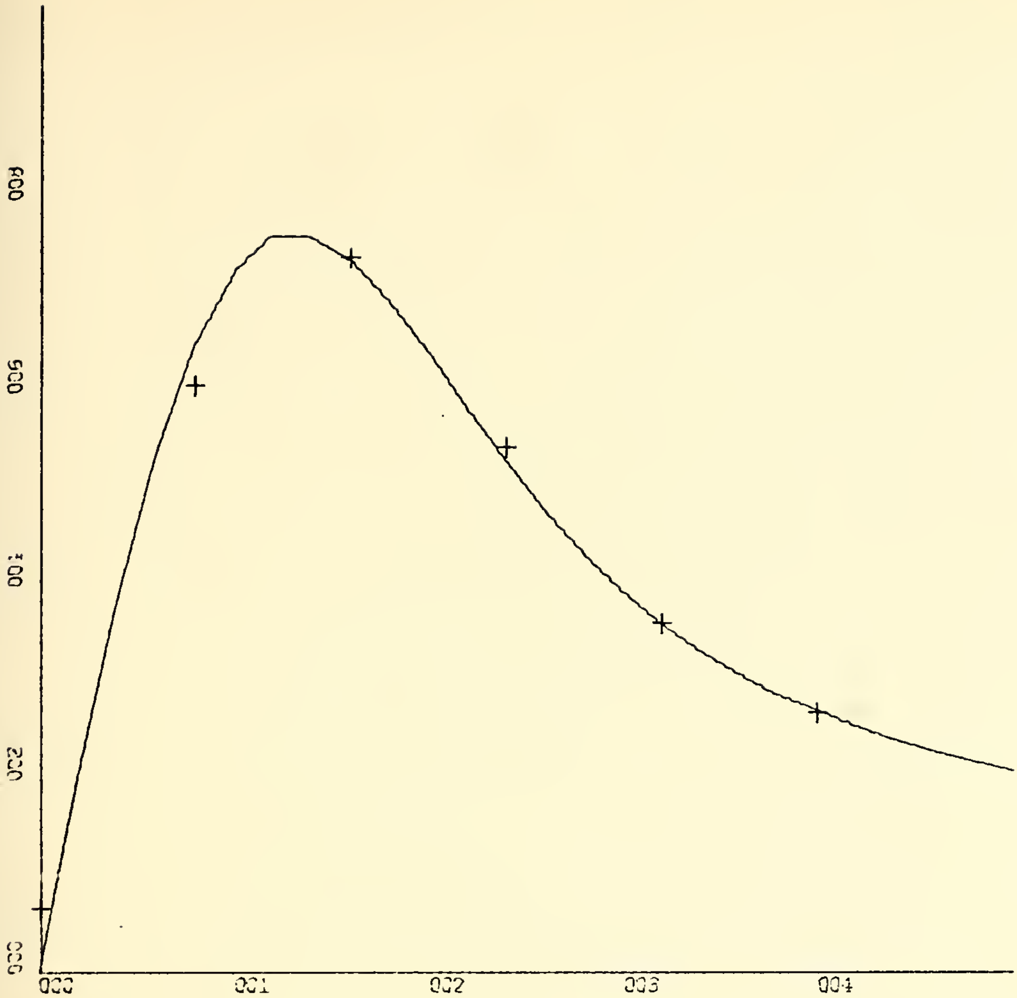
FIGURE 25



FOR DELTA T = 0.7 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.05833 |
| 0.7 | 0.56225 |
| 1.4 | 0.75470 |
| 2.1 | 0.62530 |
| 2.8 | 0.43722 |
| 3.5 | 0.31854 |
| 4.2 | 0.25438 |

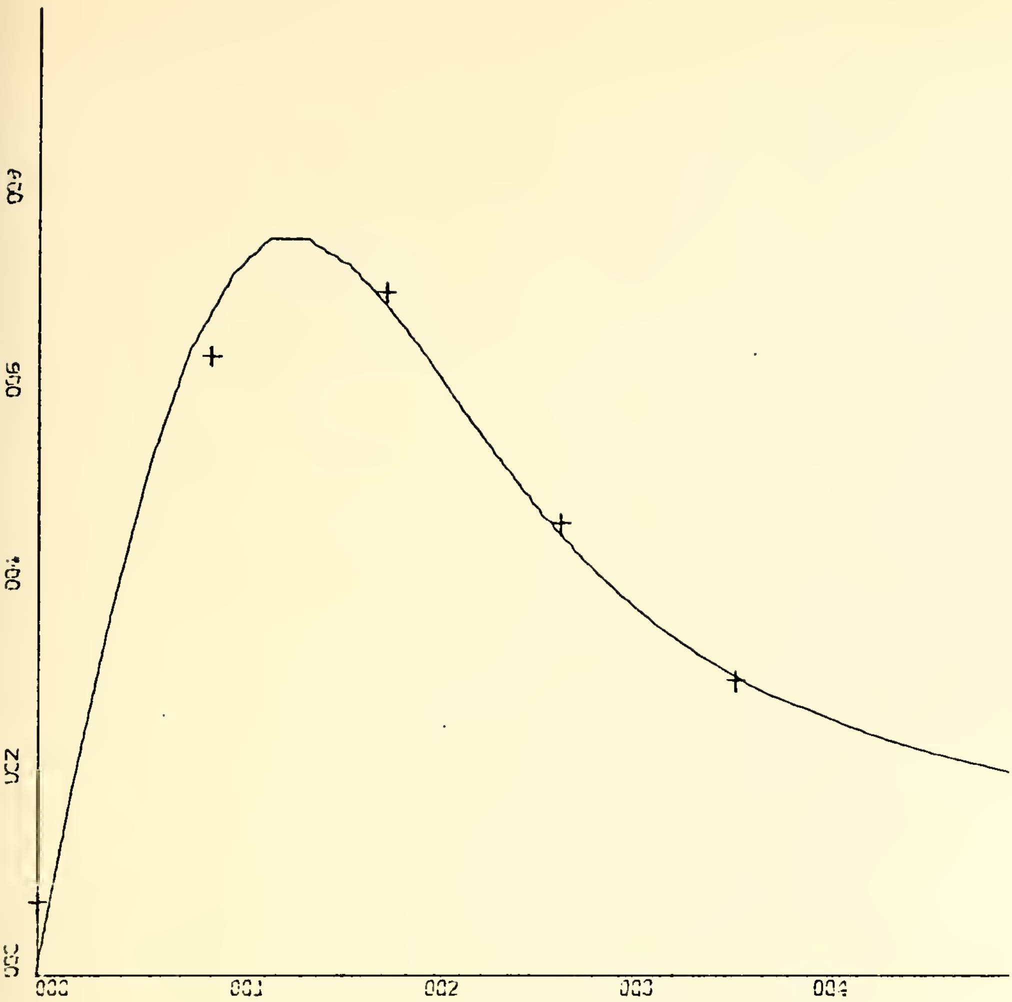
FIGURE 26



FOR DELTA T = 0.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.06667 |
| 0.8 | 0.60606 |
| 1.6 | 0.73910 |
| 2.4 | 0.54392 |
| 3.2 | 0.36042 |
| 4.0 | 0.26888 |

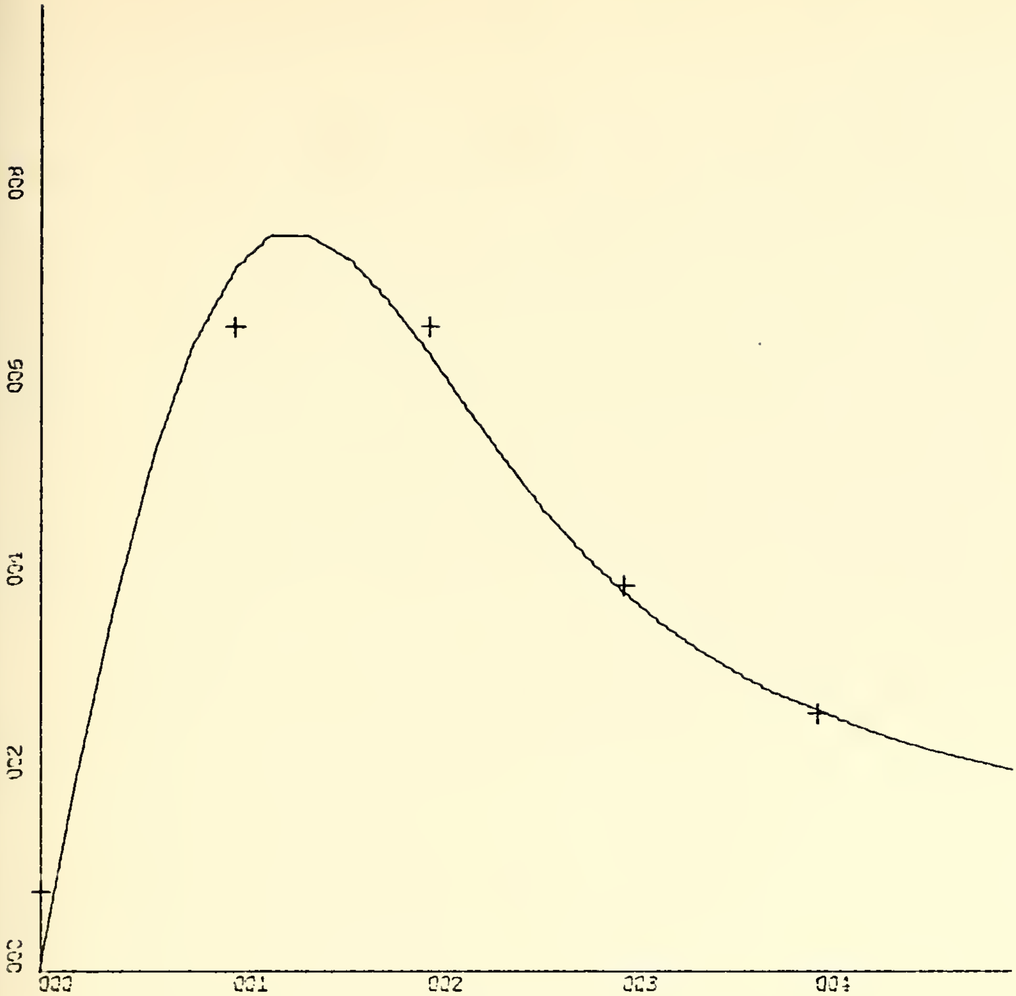
FIGURE 27



FOR DELTA T = 0.9 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.07500 |
| 0.9 | 0.64057 |
| 1.8 | 0.70781 |
| 2.7 | 0.46704 |
| 3.6 | 0.30519 |

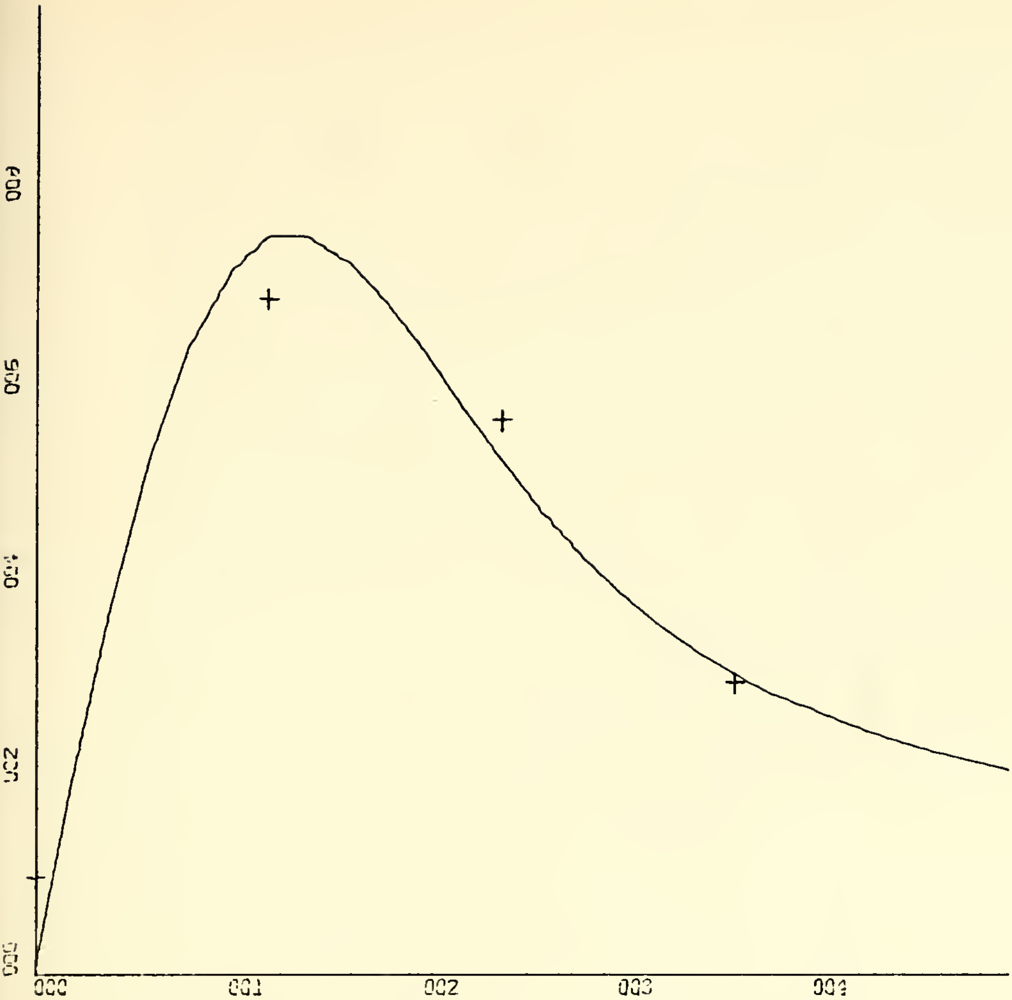
FIGURE 28



FOR DELTA T = 1.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.08333 |
| 1.0 | 0.66667 |
| 2.0 | 0.66667 |
| 3.0 | 0.40000 |
| 4.0 | 0.26667 |

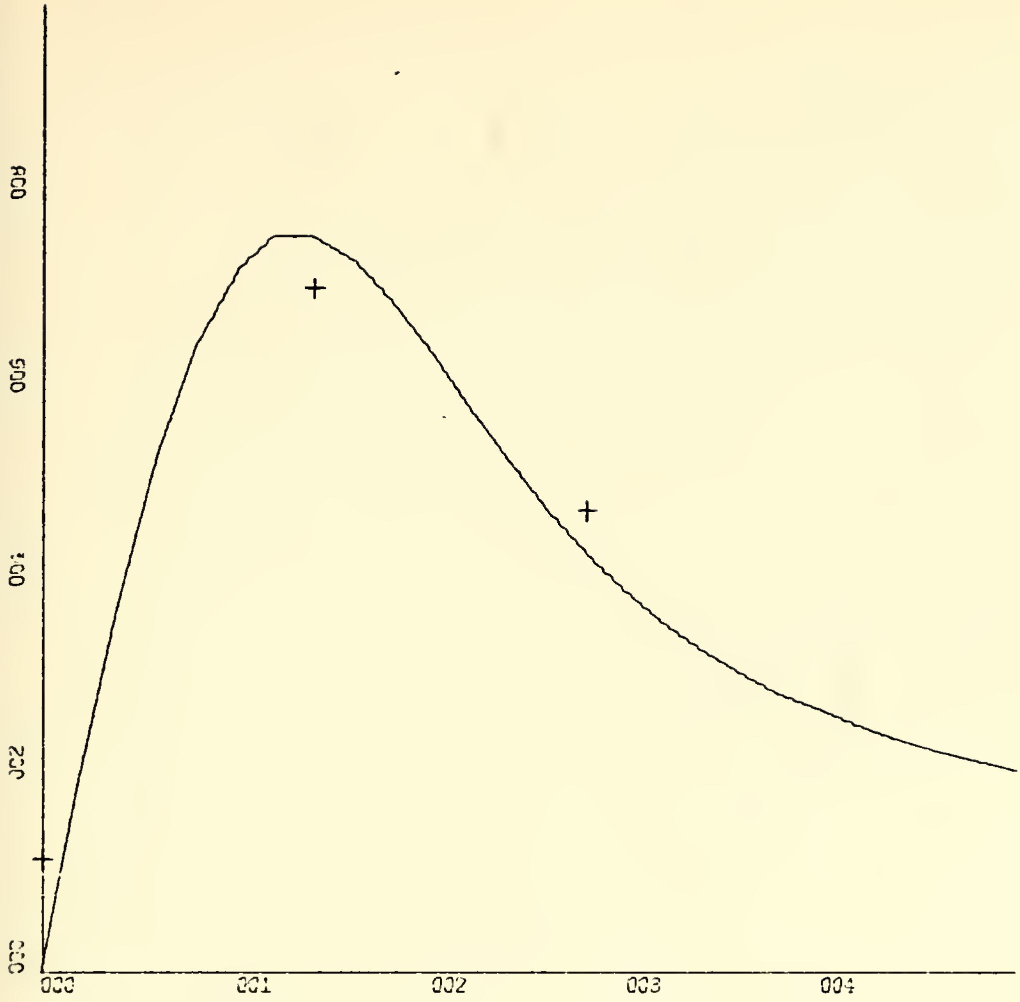
FIGURE 29



FOR DELTA T = 1.2    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.10000 |
| 1.2 | 0.69767 |
| 2.4 | 0.57186 |
| 3.6 | 0.30012 |

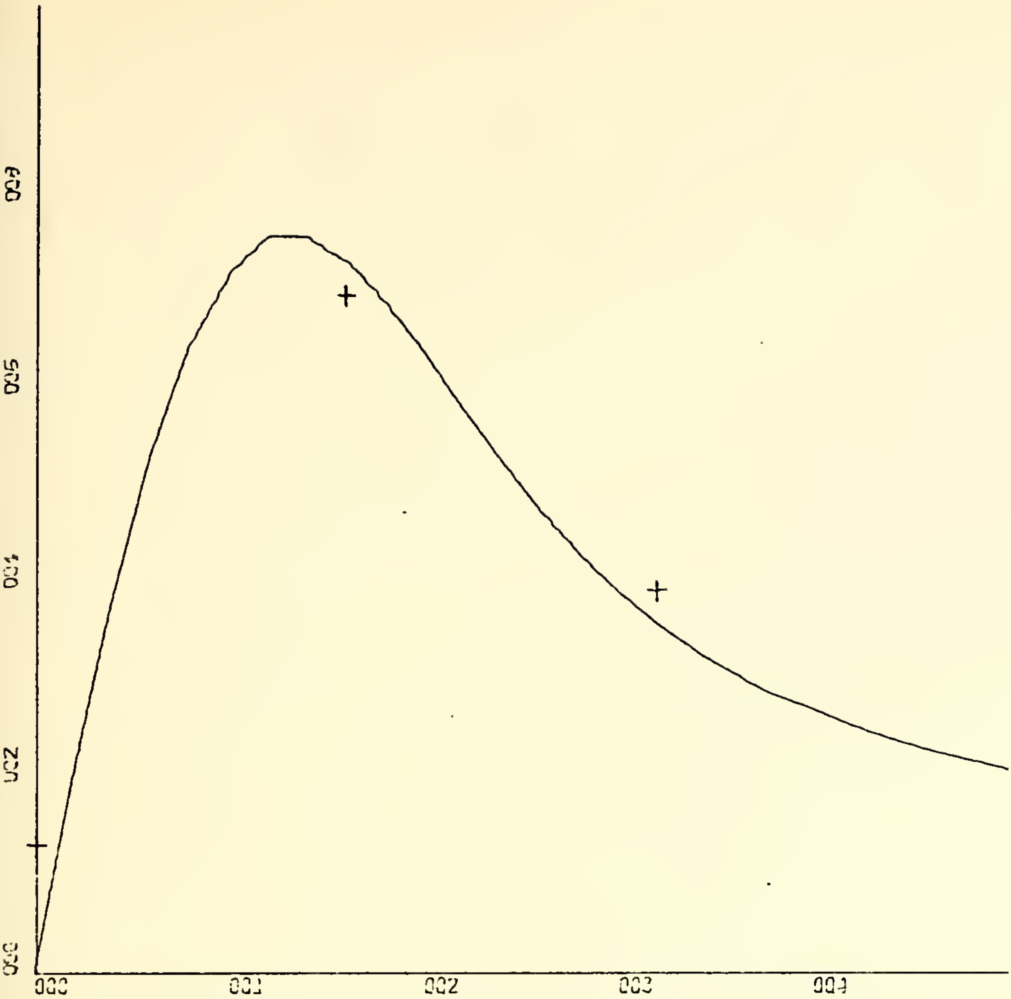
FIGURE 30



FOR DELTA T = 1.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.11667 |
| 1.4 | 0.70707 |
| 2.8 | 0.47775 |

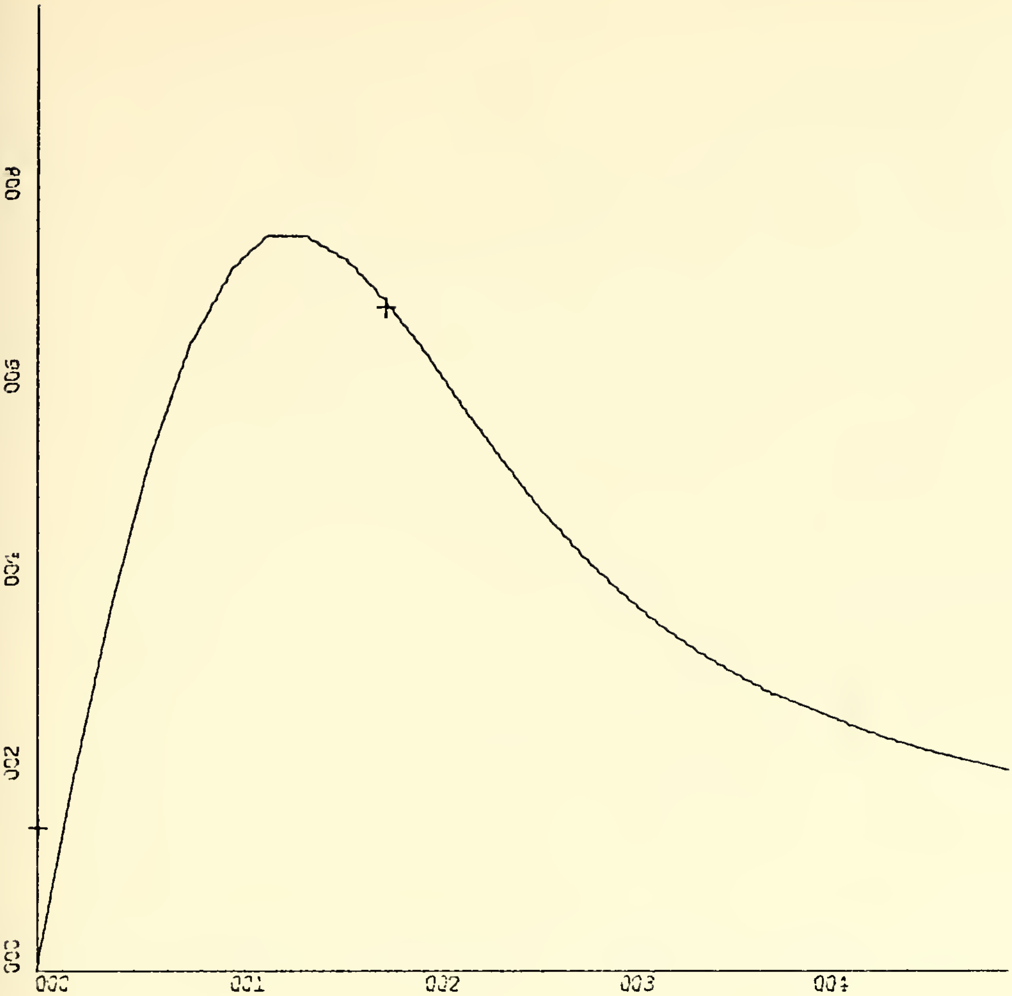
FIGURE 31



FOR DELTA T = 1.6     TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.13333 |
| 1.6 | 0.70175 |
| 3.2 | 0.39424 |

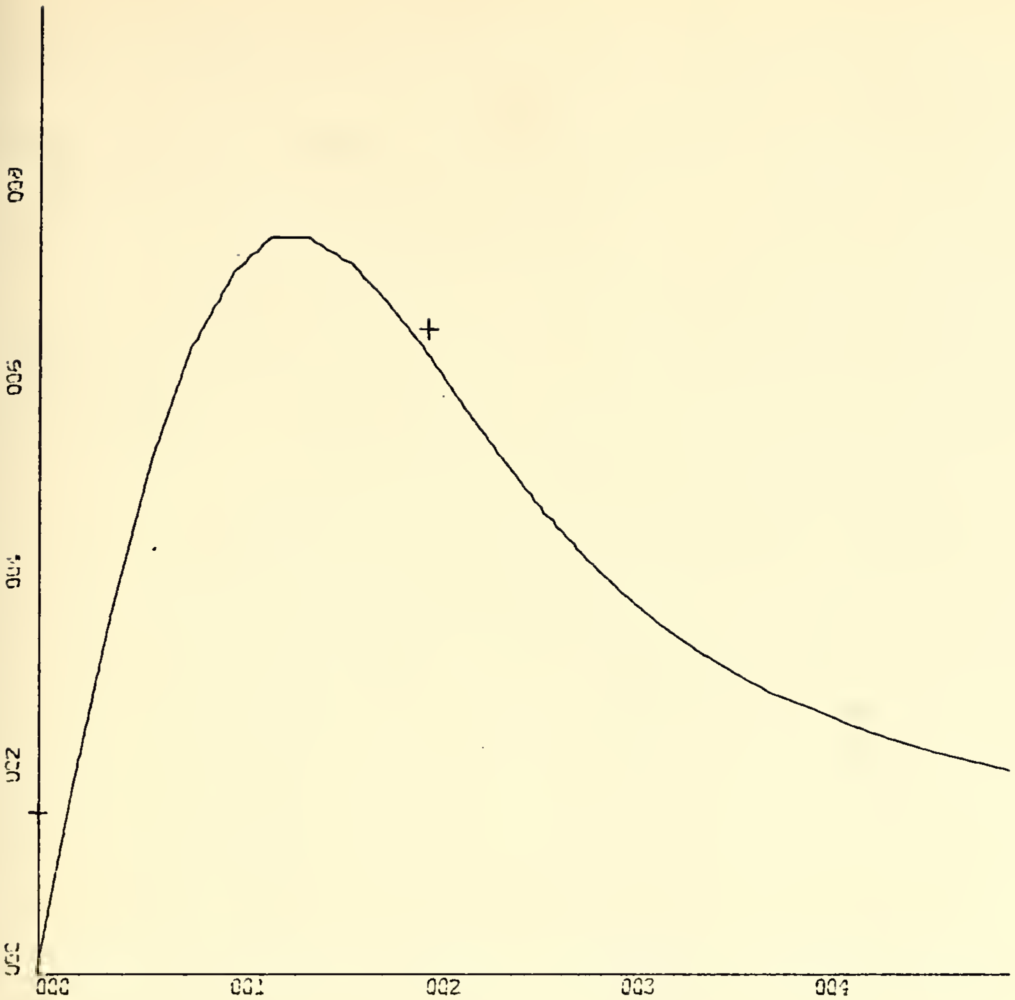
FIGURE 32



FOR DELTA T = 1.8      TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.15000 |
| 1.8 | 0.68702 |

FIGURE 33



FOR DELTA T = 2.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.16667 |
| 2.0 | 0.66667 |

FIGURE 34

2. Case 3 — $\dot{Y} + t^2 Y = 1$

As in Case 1 and Case 2, the same calculations are followed and only in the division, the substitution of $P(t) = t^2$ is the sole difference.

Output for several time iterations follows as in the previous cases (Figs. 35 to 48). Computer algorithm is shown in Program 3 (page 63).

FOR DELTA T = 0.2 TFIN = 5.2

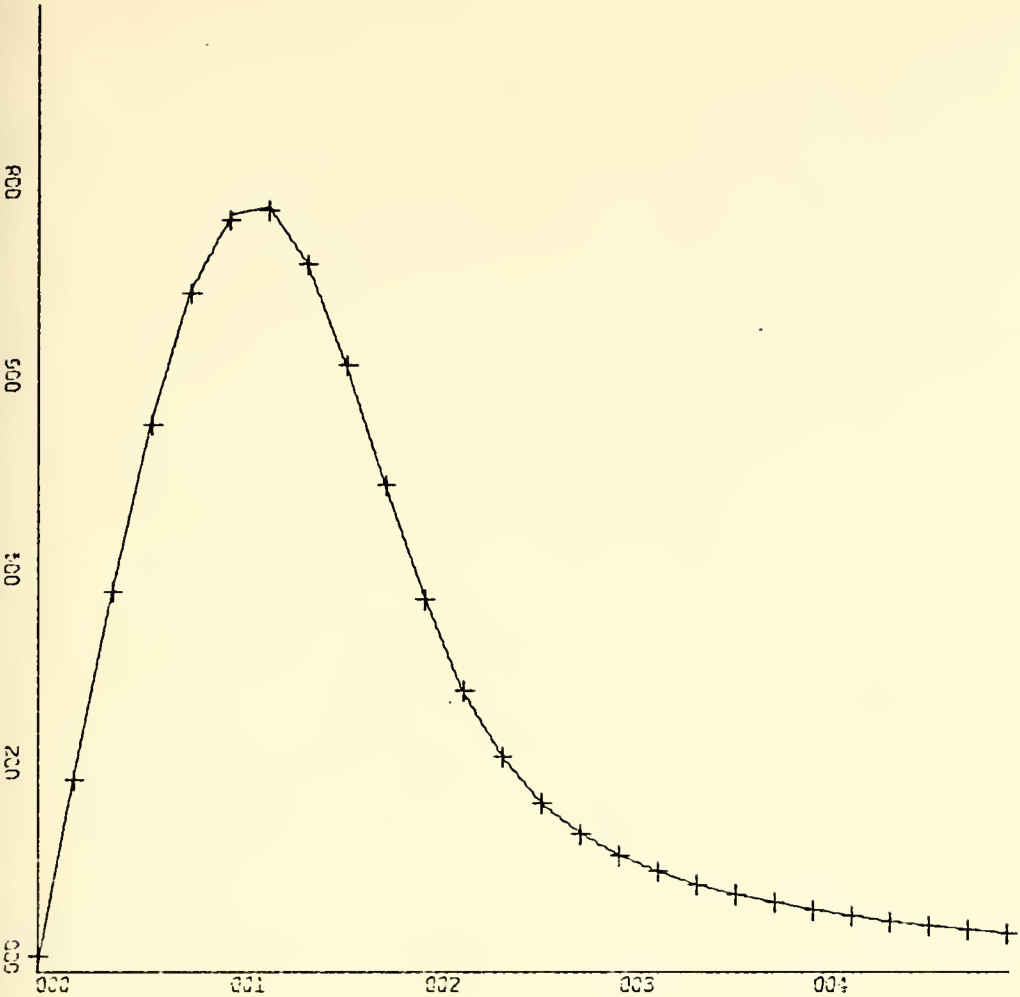
| T | Q(N) | PR |
|---|---|---|
| 0.0 | 0.01667 | 0.0 |
| 0.2 | 0.19920 | 0.19960 |
| 0.4 | 0.39213 | 0.39366 |
| 0.6 | 0.56550 | 0.56858 |
| 0.8 | 0.70032 | 0.70472 |
| 1.0 | 0.77773 | 0.78240 |
| 1.2 | 0.78667 | 0.79011 |
| 1.4 | 0.73026 | 0.73132 |
| 1.6 | 0.62669 | 0.62542 |
| 1.8 | 0.50321 | 0.50080 |
| 2.0 | 0.38584 | 0.38376 |
| 2.2 | 0.29077 | 0.28980 |
| 2.4 | 0.22210 | 0.22208 |
| 2.6 | 0.17552 | 0.17589 |
| 2.8 | 0.14398 | 0.14433 |
| 3.0 | 0.12163 | 0.12183 |
| 3.2 | 0.10482 | 0.10492 |
| 3.4 | 0.09160 | 0.09165 |
| 3.6 | 0.08088 | 0.08091 |
| 3.8 | 0.07204 | 0.07205 |
| 4.0 | 0.06462 | 0.06464 |
| 4.2 | 0.05833 | 0.05834 |
| 4.4 | 0.05294 | 0.05295 |
| 4.6 | 0.04828 | 0.04830 |
| 4.8 | 0.04422 | 0.04423 |
| 5.0 | 0.04067 | 0.04067 |

Equation (45) is repeated here for convenience:

$$y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=STEP  , P(T)=T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = T**2
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C     Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,7) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,7) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('   XX=',7X,'    PR=')
      END
```

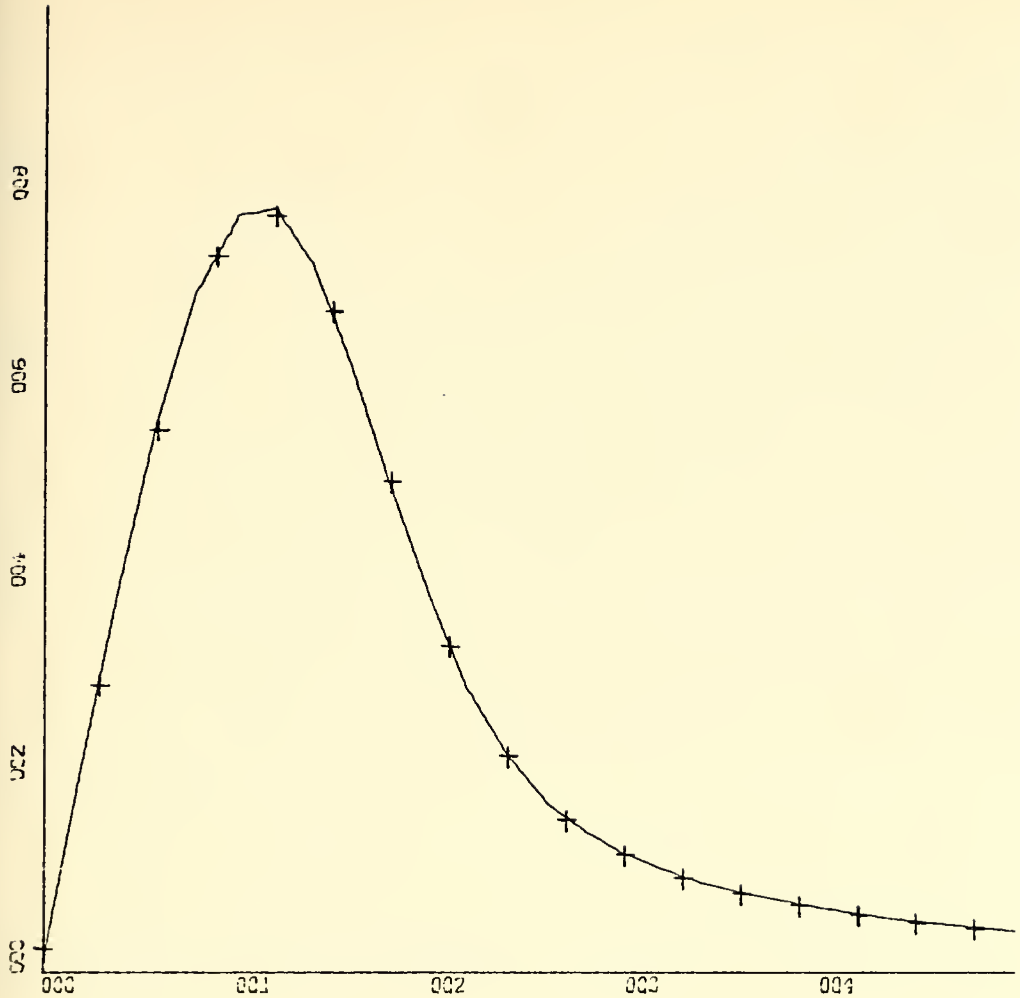
PROGRAM 3



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

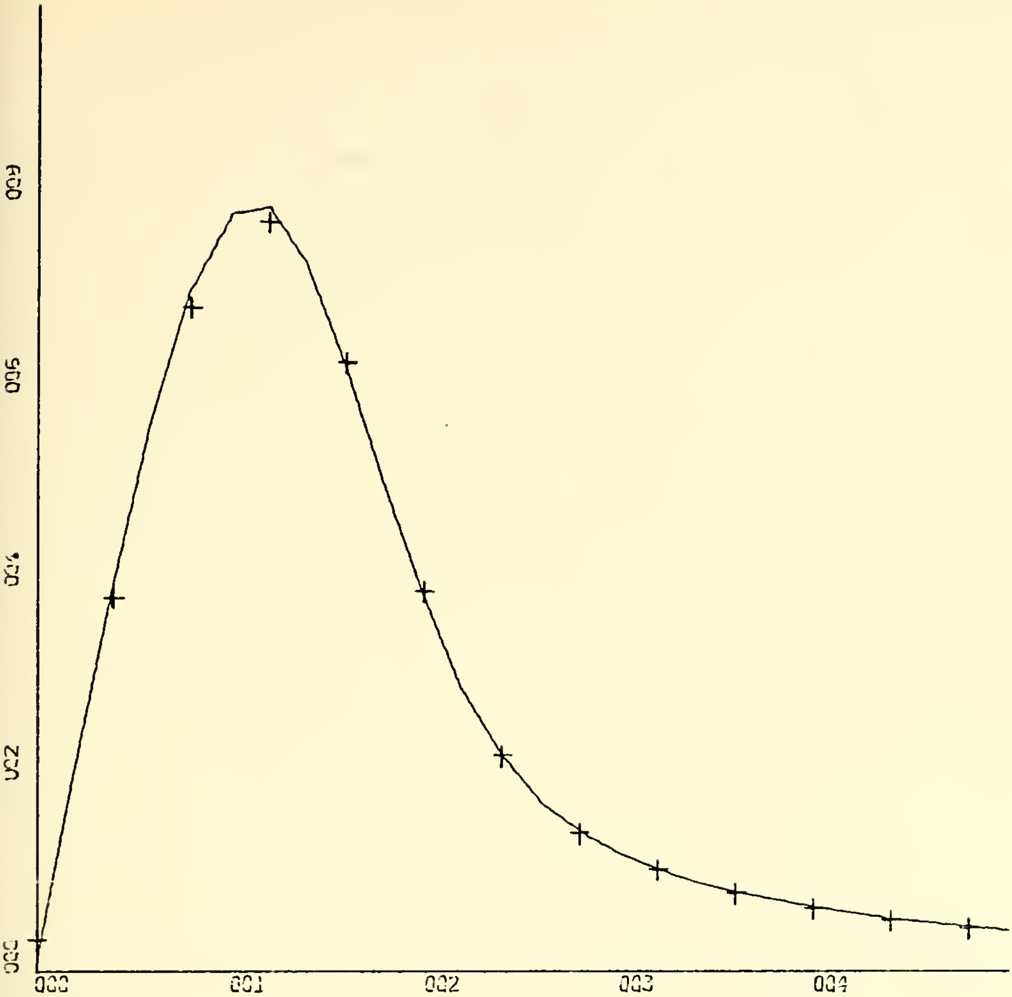
FIGURE 35



FOR DELTA T = 0.3 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.02500 |
| 0.3 | 0.29600 |
| 0.6 | 0.56168 |
| 0.9 | 0.74128 |
| 1.2 | 0.78225 |
| 1.5 | 0.68283 |
| 1.8 | 0.50631 |
| 2.1 | 0.33719 |
| 2.4 | 0.22218 |
| 2.7 | 0.15773 |
| 3.0 | 0.12138 |
| 3.3 | 0.09778 |
| 3.6 | 0.08086 |
| 3.9 | 0.06816 |
| 4.2 | 0.05832 |
| 4.5 | 0.05053 |
| 4.8 | 0.04422 |

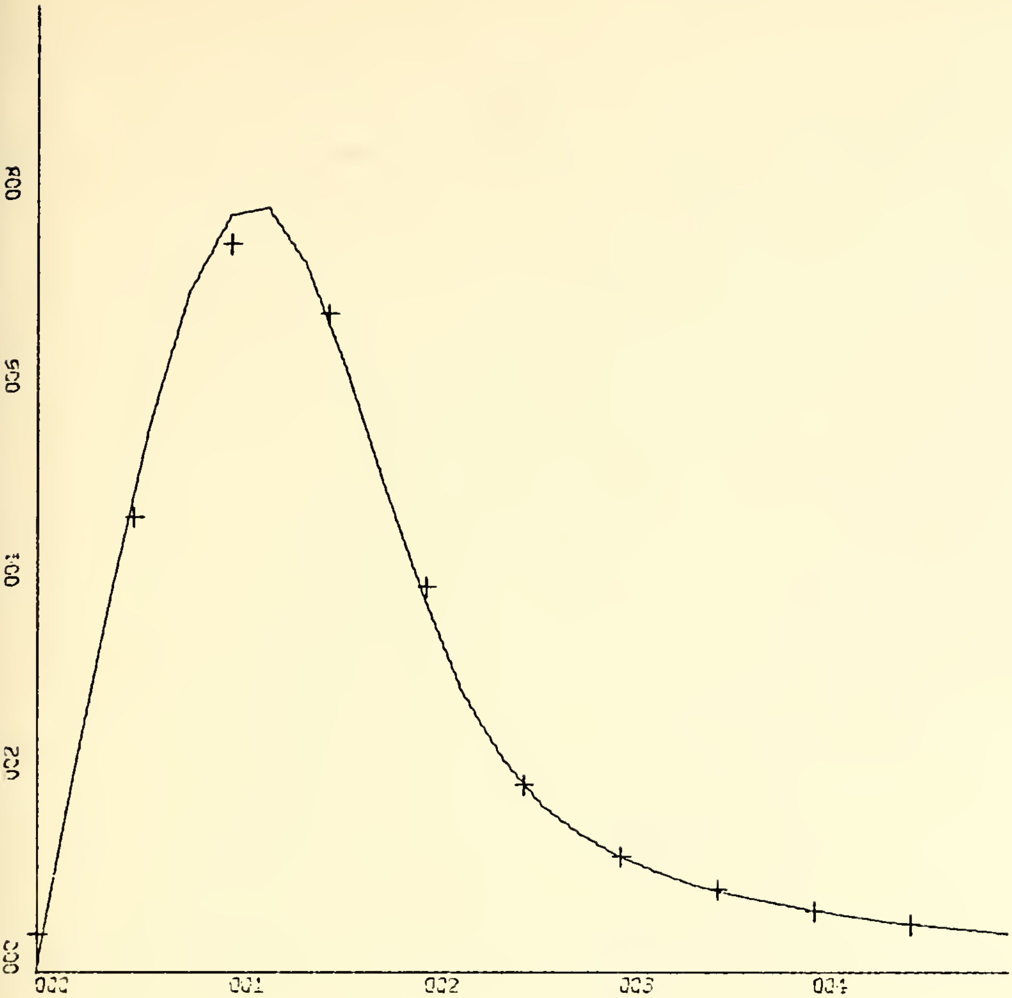
FIGURE 36



FOR DELTA T = 0.4    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.03333 |
| 0.4 | 0.38760 |
| 0.8 | 0.68723 |
| 1.2 | 0.77582 |
| 1.6 | 0.62989 |
| 2.0 | 0.39299 |
| 2.4 | 0.22240 |
| 2.8 | 0.14260 |
| 3.2 | 0.10466 |
| 3.6 | 0.08082 |
| 4.0 | 0.06460 |
| 4.4 | 0.05293 |
| 4.8 | 0.04422 |

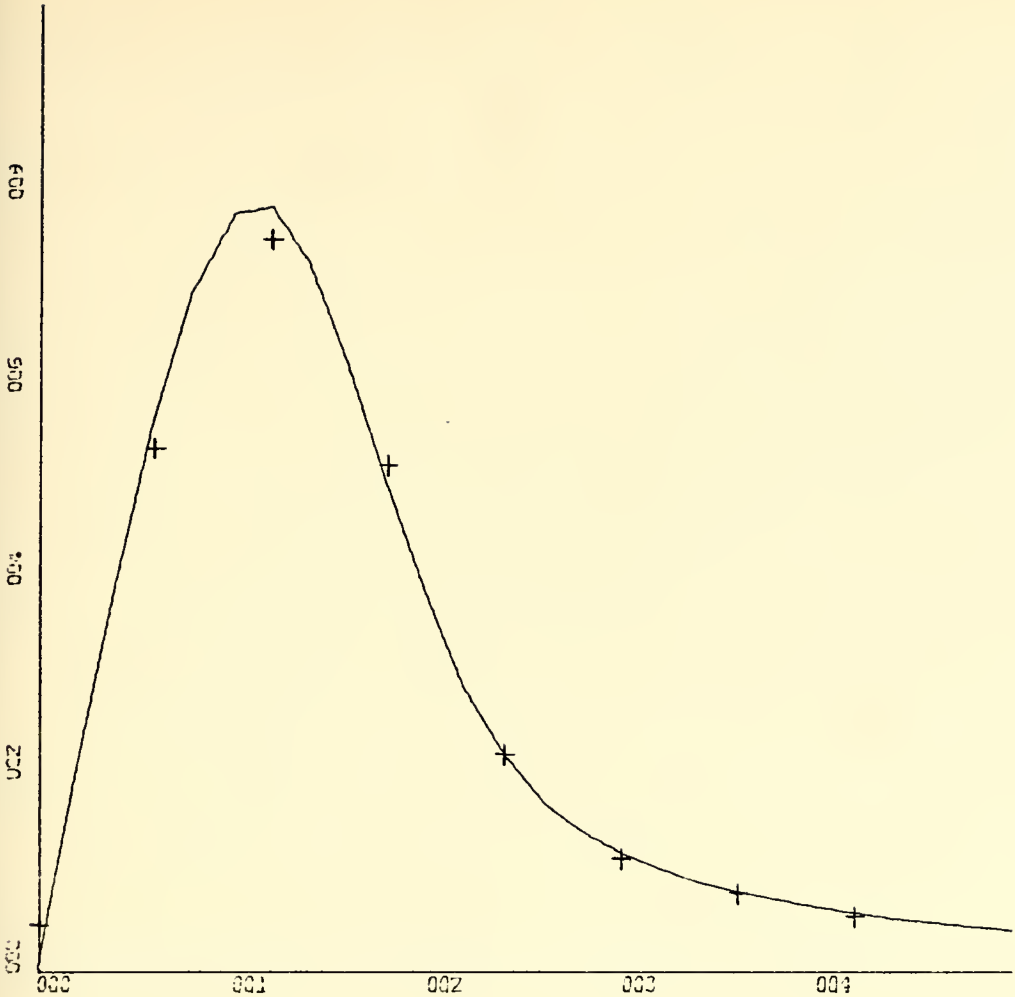
FIGURE 37



FOR DELTA T = 0.5 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.04167 |
| 0.5 | 0.47059 |
| 1.0 | 0.75294 |
| 1.5 | 0.68141 |
| 2.0 | 0.39906 |
| 2.5 | 0.19512 |
| 3.0 | 0.12007 |
| 3.5 | 0.08613 |
| 4.0 | 0.06447 |
| 4.5 | 0.05057 |

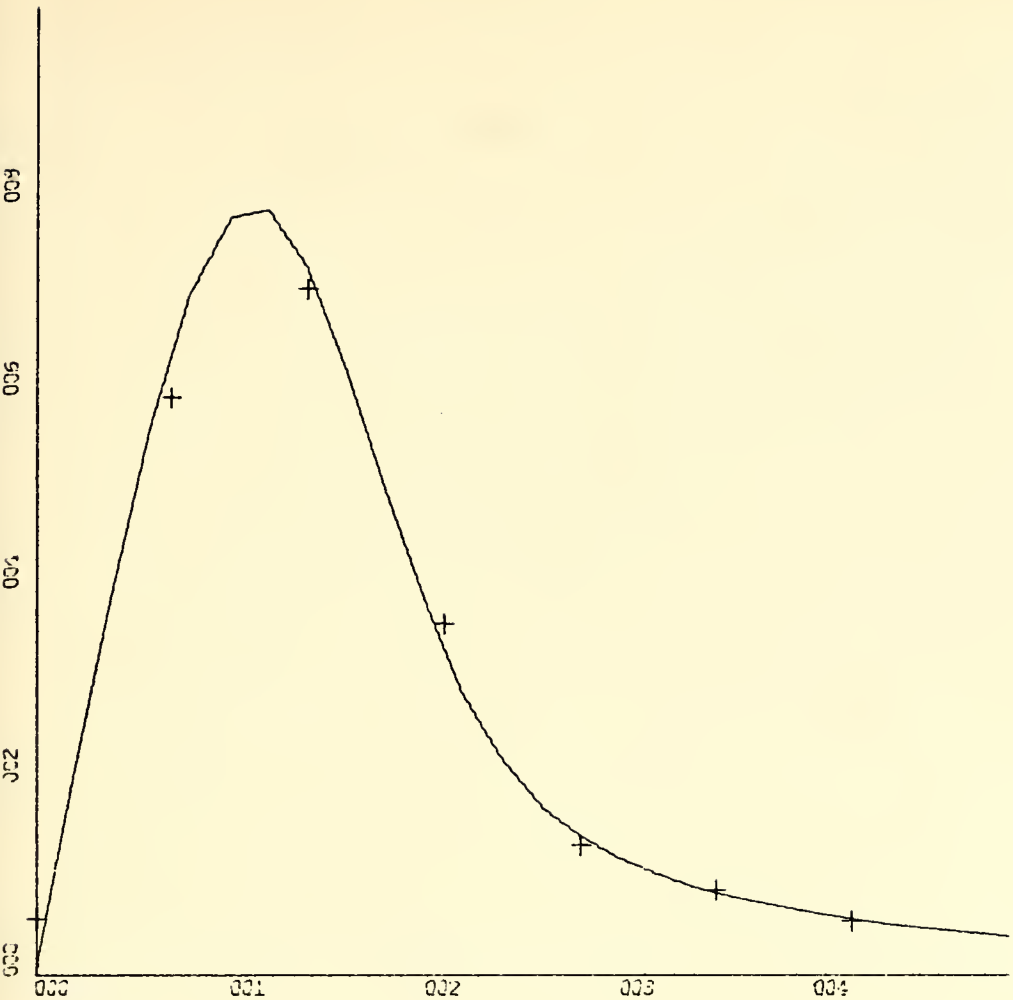
FIGURE 38



FOR DELTA T = 0.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.05 |
| 0.6 | 0.54152 |
| 1.2 | 0.75631 |
| 1.8 | 0.52210 |
| 2.4 | 0.22530 |
| 3.0 | 0.11783 |
| 3.6 | 0.08177 |
| 4.2 | 0.05783 |

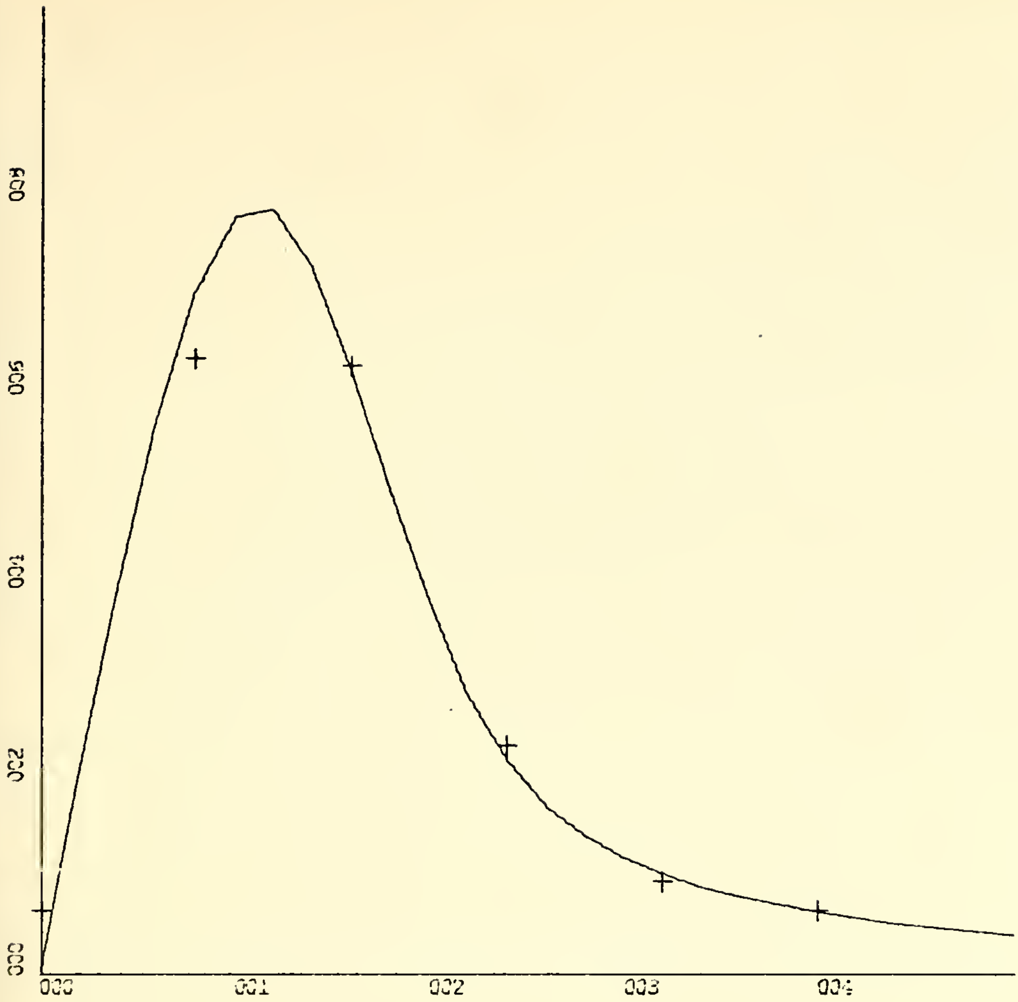
FIGURE 39



FOR DELTA T = 0.7 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.05833 |
| 0.7 | 0.59752 |
| 1.4 | 0.70881 |
| 2.1 | 0.36272 |
| 2.8 | 0.13431 |
| 3.5 | 0.08809 |
| 4.2 | 0.05721 |

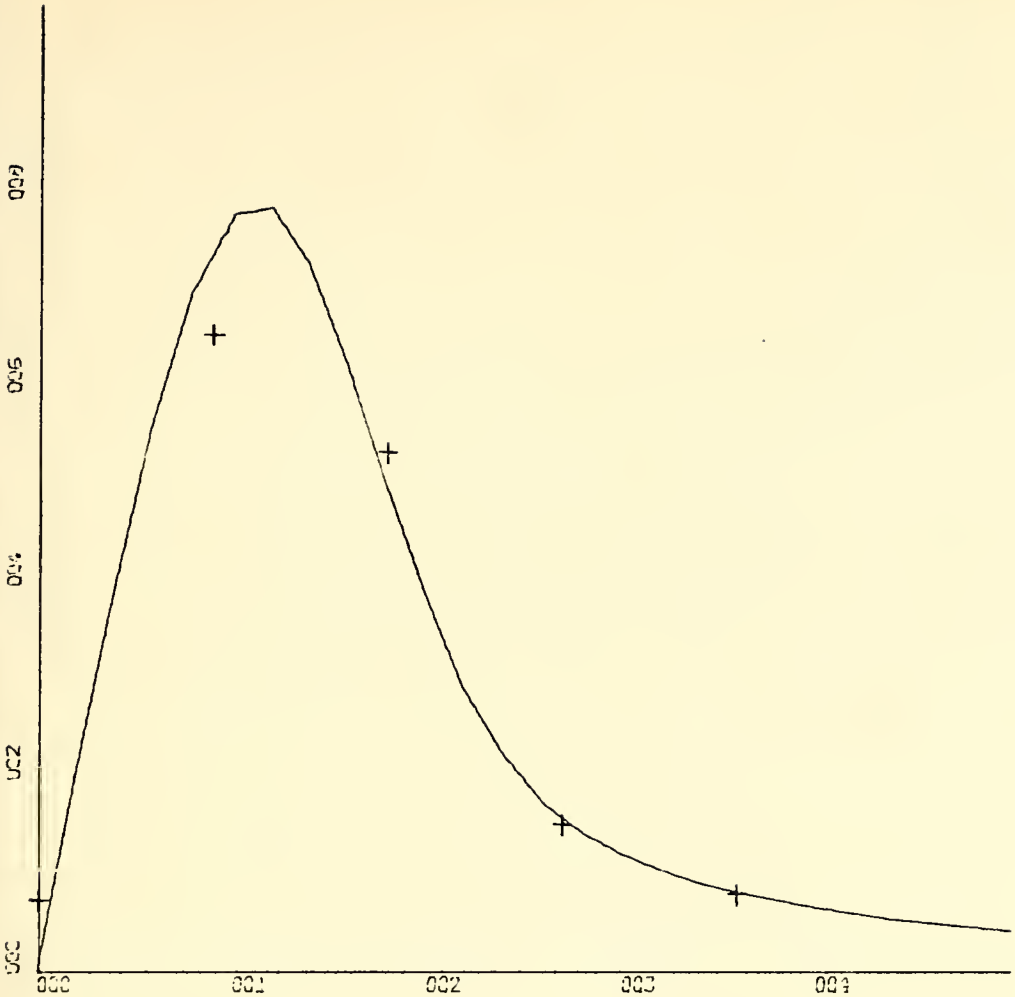
FIGURE 40




FOR DELTA T = 0.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.06667 |
| 0.8 | 0.63694 |
| 1.6 | 0.62939 |
| 2.4 | 0.23756 |
| 3.2 | 0.09620 |
| 4.0 | 0.06786 |

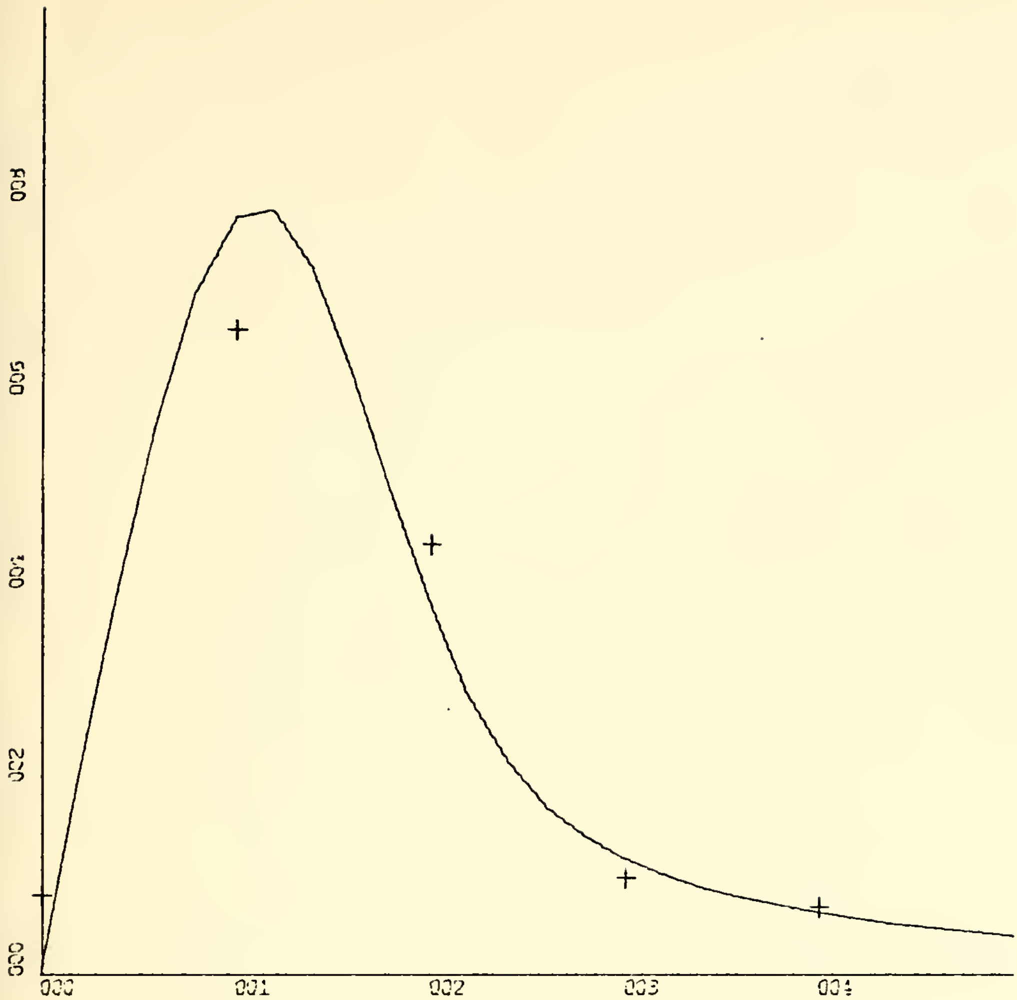
FIGURE 41



FOR DELTA T = 0.9    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.07500 |
| 0.9 | 0.65958 |
| 1.8 | 0.53668 |
| 2.7 | 0.15283 |
| 3.6 | 0.08072 |

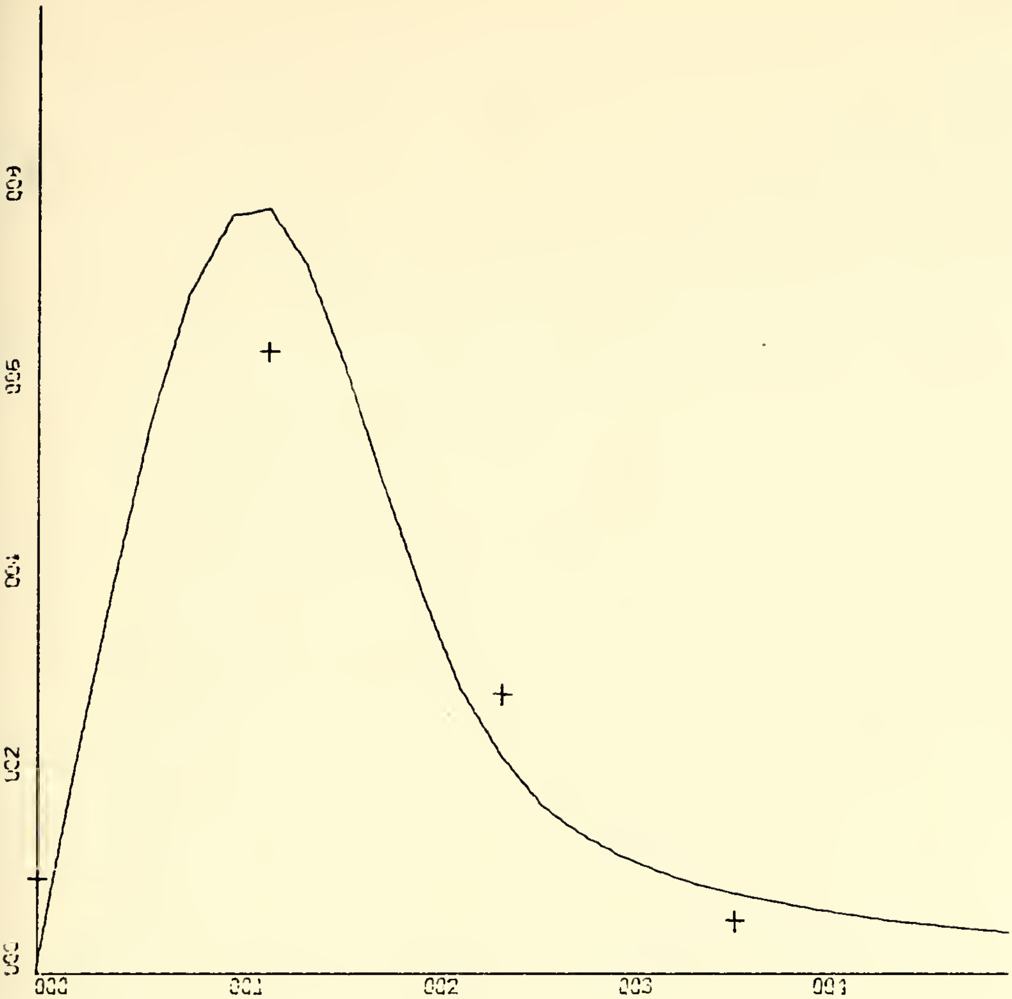
FIGURE 42



FOR DELTA T = 1.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.08333 |
| 1.0 | 0.66667 |
| 2.0 | 0.44444 |
| 3.0 | 0.10101 |
| 4.0 | 0.07183 |

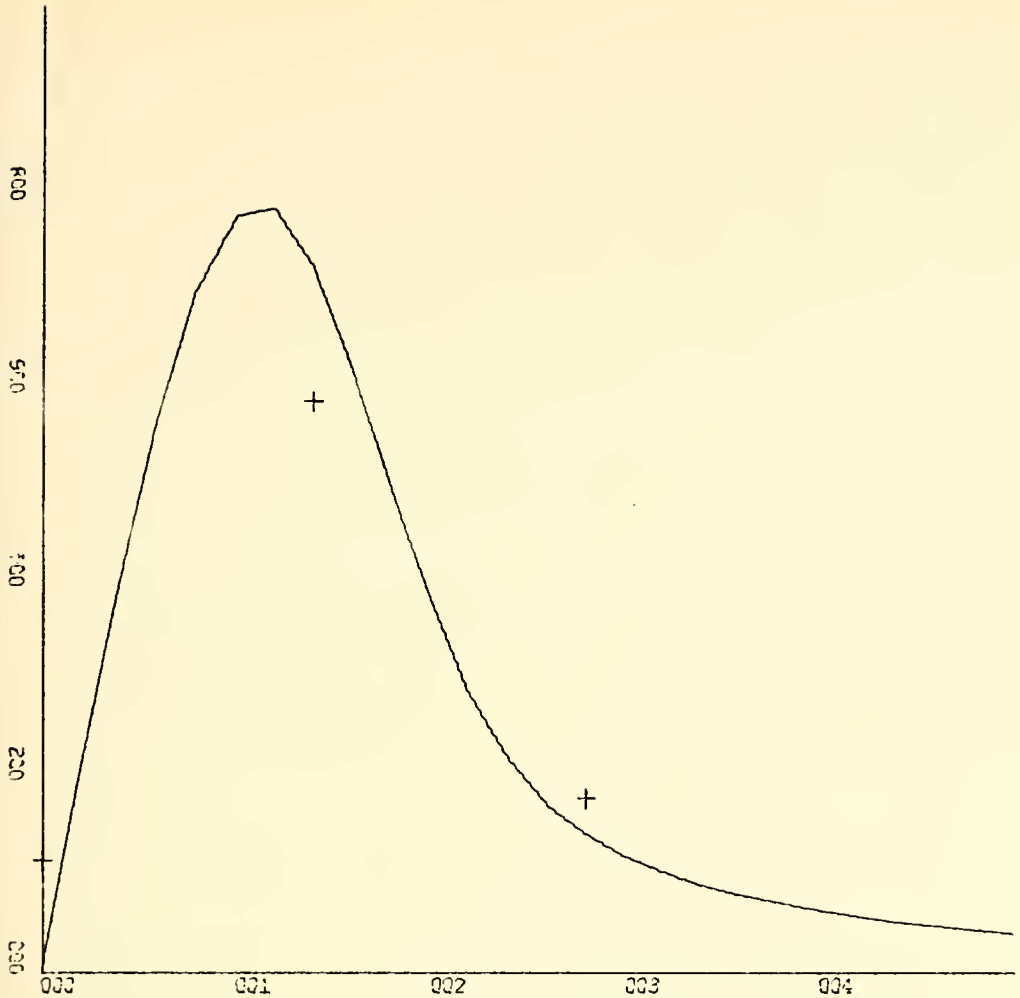
FIGURE 43



FOR DELTA T = 1.2 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.10000 |
| 1.2 | 0.64378 |
| 2.4 | 0.28895 |
| 3.6 | 0.05587 |

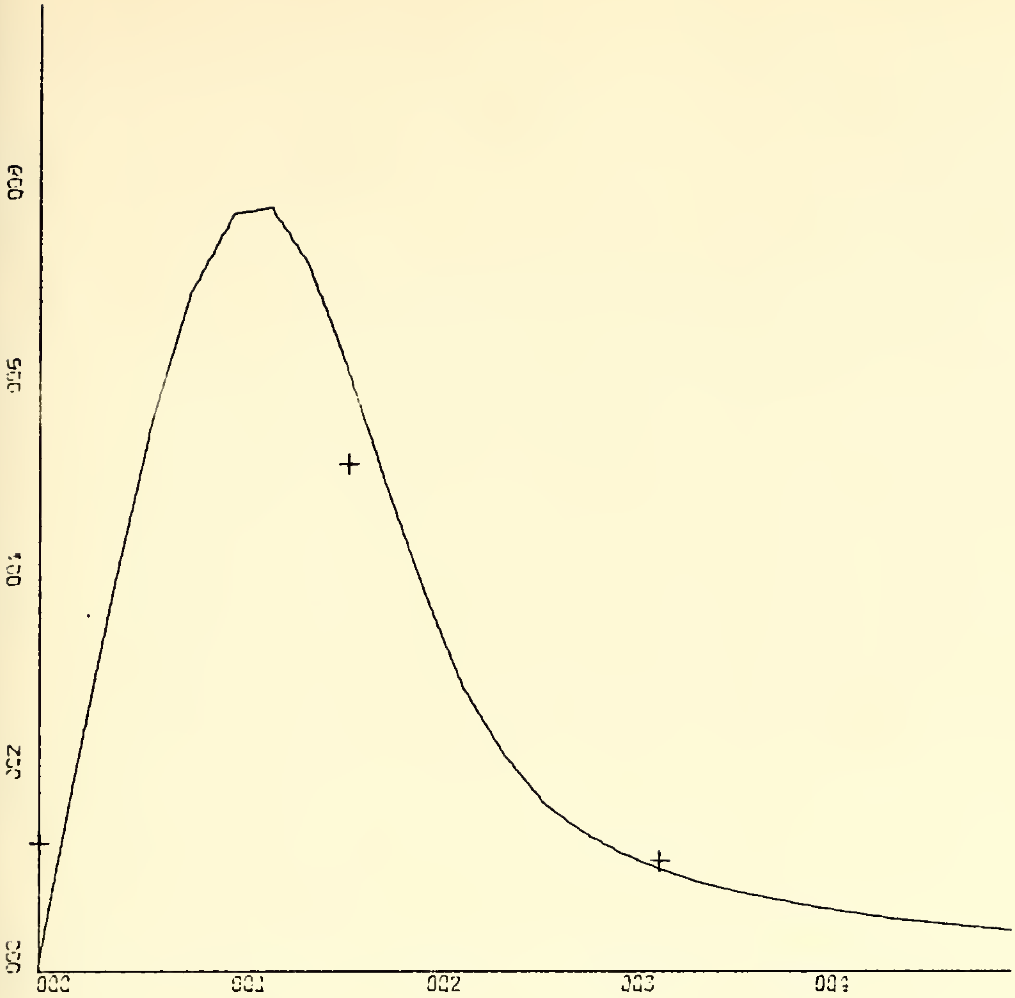
FIGURE 44



FOR DELTA T = 1.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.11667 |
| 1.4 | 0.59022 |
| 2.8 | 0.18194 |

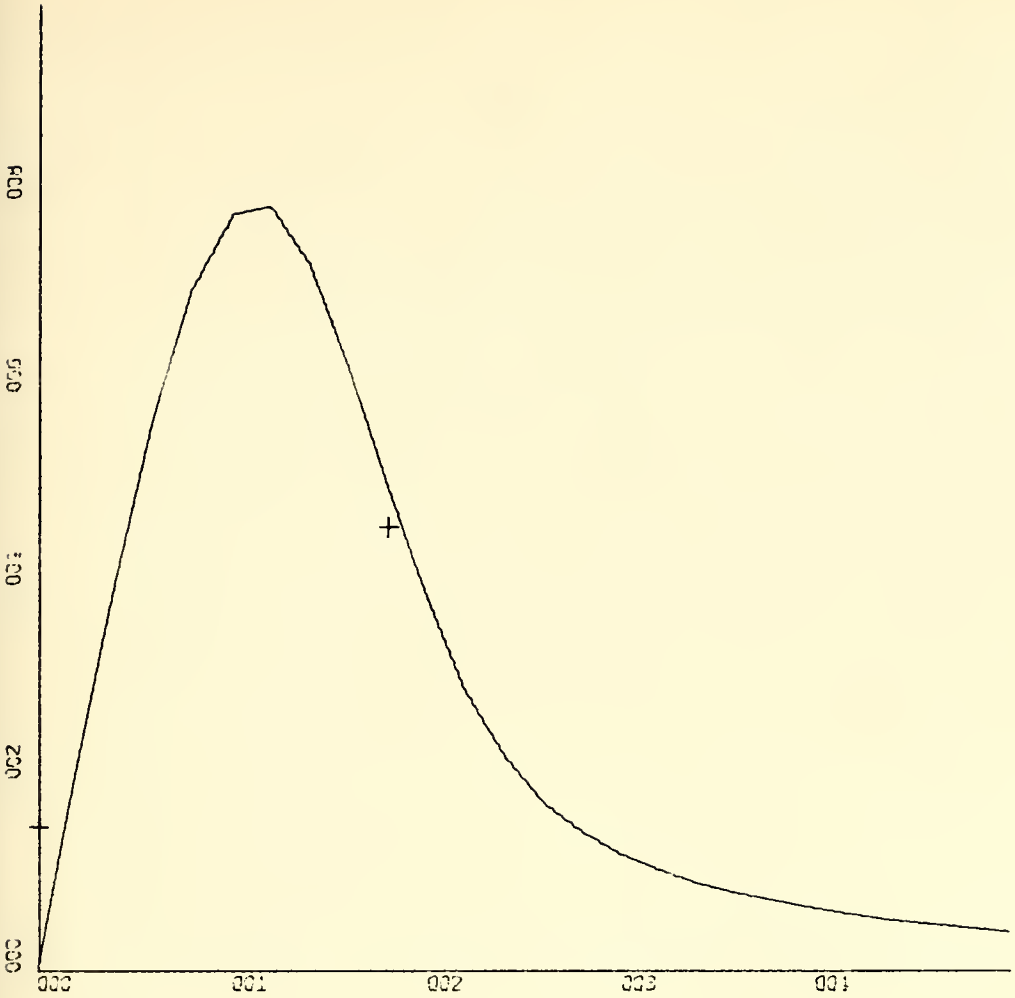
FIGURE 45



FOR DELTA T = 1.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.13333 |
| 1.6 | 0.52493 |
| 3.2 | 0.11422 |

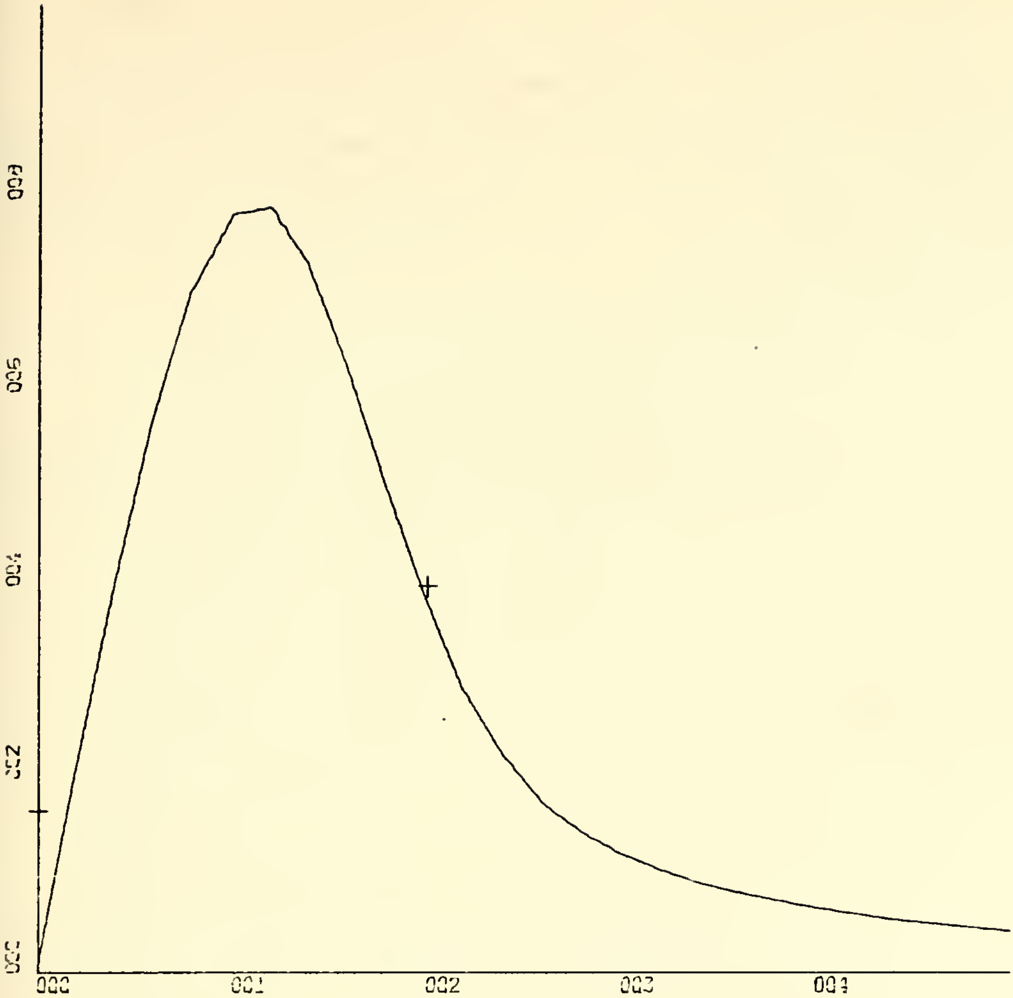
FIGURE 46



FOR DELTA T = 1.8    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.15000 |
| 1.8 | 0.45965 |

FIGURE 47



FOR DELTA T = 2.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.0 | 0.16667 |
| 2.0 | 0.40000 |

FIGURE 48

C. CASE 4 — $\dot{Y} + \sqrt{t}\,Y = t$

For this case, in the equation $\dot{Y} + P(t)Y = Q(t)$, $P(t) = \sqrt{t}$ and $Q(t) = t$, so following the procedure the equation is rewritten as:

$$\frac{dY}{dt} + PY = t \qquad Y_{(t=0)} = 0$$

where $P = \sqrt{t}$ . The Laplace transform is:

$$sy + Py = 1/s^2$$

so

$$s^3y + Ps^2y = 1 \qquad y(s^3 + Ps^2) = 1$$

$$y = \frac{1}{s^3 + Ps^2} = \frac{s^{-3}}{1 + Ps^{-1}}$$

Substituting the z-forms of $s^{-k}$ from Table I one gets:

$$y_A{*}(z) = \frac{\dfrac{T^3}{2}\,\dfrac{z^{-1} + z^{-2}}{(1 - z^{-1})^3}}{1 + P\,\dfrac{T}{2}\,\dfrac{1 + z^{-1}}{1 - z^{-1}}}$$

Dividing by T and ordering in crescent powers of $z^{-1}$ in numerator and denominator:

$$y_A{*}(z) = \frac{T^2z^{-1} + T^2z^{-2}}{(2+Pt) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}} \tag{46}$$

Now, substituting for the time iteration wanted and replacing $P = \sqrt{t}$ for its value at the division step, the solution is obtained.

In the same way as before, plots and outputs follow for time iterations from 0.2 sec. to 2.0 sec. (Figs. 49 to 60).

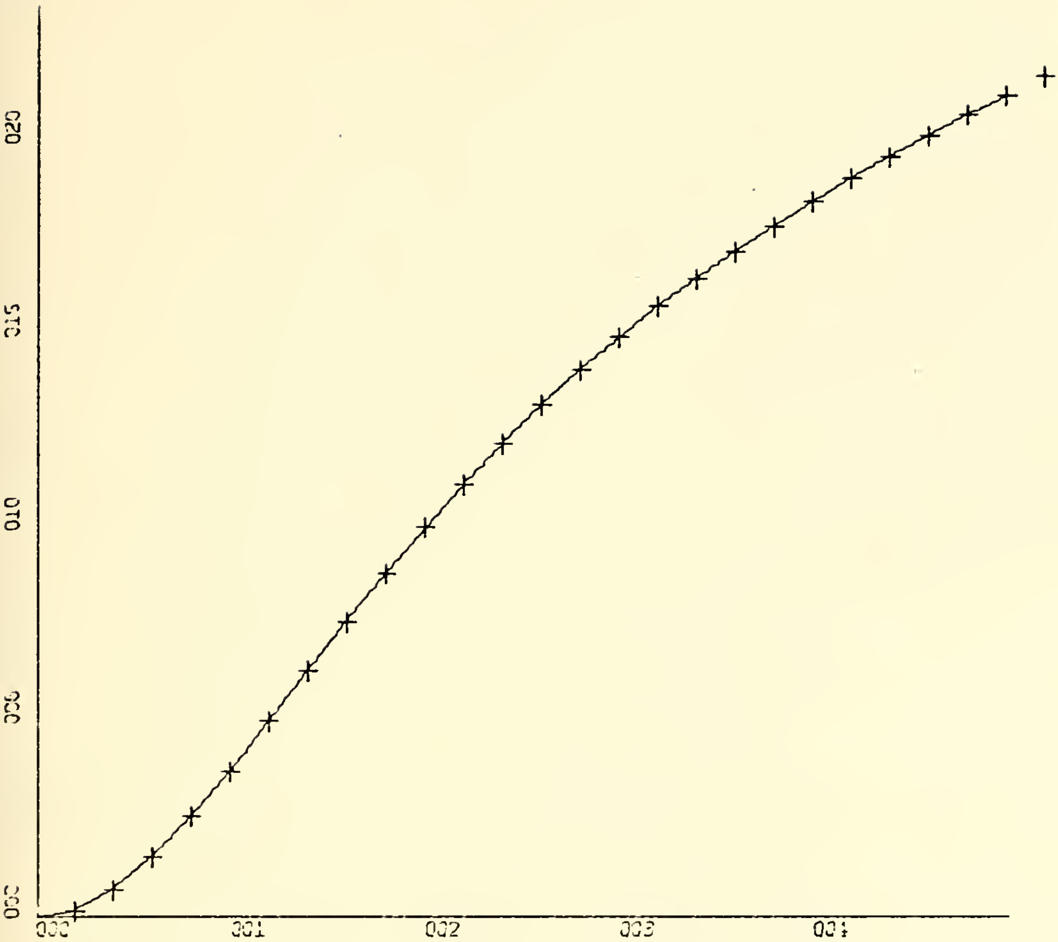
Algorithm for computer solution of this problem is shown in Program 4 (page 80).

FOR DELTA T = 0.2 TFIN = 5.2

| T | Q(N) | PR |
|---|---|---|
| 0.2 | 0.01914 | 0.0 |
| 0.4 | 0.07363 | 0.01949 |
| 0.6 | 0.15683 | 0.07449 |
| 0.8 | 0.26131 | 0.15817 |
| 1.0 | 0.37994 | 0.26299 |
| 1.2 | 0.50647 | 0.38181 |
| 1.4 | 0.63576 | 0.50837 |
| 1.6 | 0.76391 | 0.63756 |
| 1.8 | 0.88813 | 0.76551 |
| 2.0 | 1.00662 | 0.88948 |
| 2.2 | 1.11838 | 1.00770 |
| 2.4 | 1.22302 | 1.11920 |
| 2.6 | 1.32060 | 1.22360 |
| 2.8 | 1.41147 | 1.32100 |
| 3.0 | 1.49613 | 1.41170 |
| 3.2 | 1.57520 | 1.49620 |
| 3.4 | 1.64928 | 1.57520 |
| 3.6 | 1.71898 | 1.64930 |
| 3.8 | 1.78485 | 1.71890 |
| 4.0 | 1.84739 | 1.78480 |
| 4.2 | 1.90703 | 1.84730 |
| 4.4 | 1.96413 | 1.90700 |
| 4.6 | 2.01902 | 1.96410 |
| 4.8 | 2.07196 | 2.01900 |
| 5.0 | 2.12316 | 2.07200 |
| 5.2 | 2.17281 | 2.12320 |

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC3C
C
C   Q(T)=RAMP   &   P(T)=SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100),  Q(100),  X(100),  XX(100),  PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = SQRT(T)
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('   XX=',7X,'    PR=')
      END
```

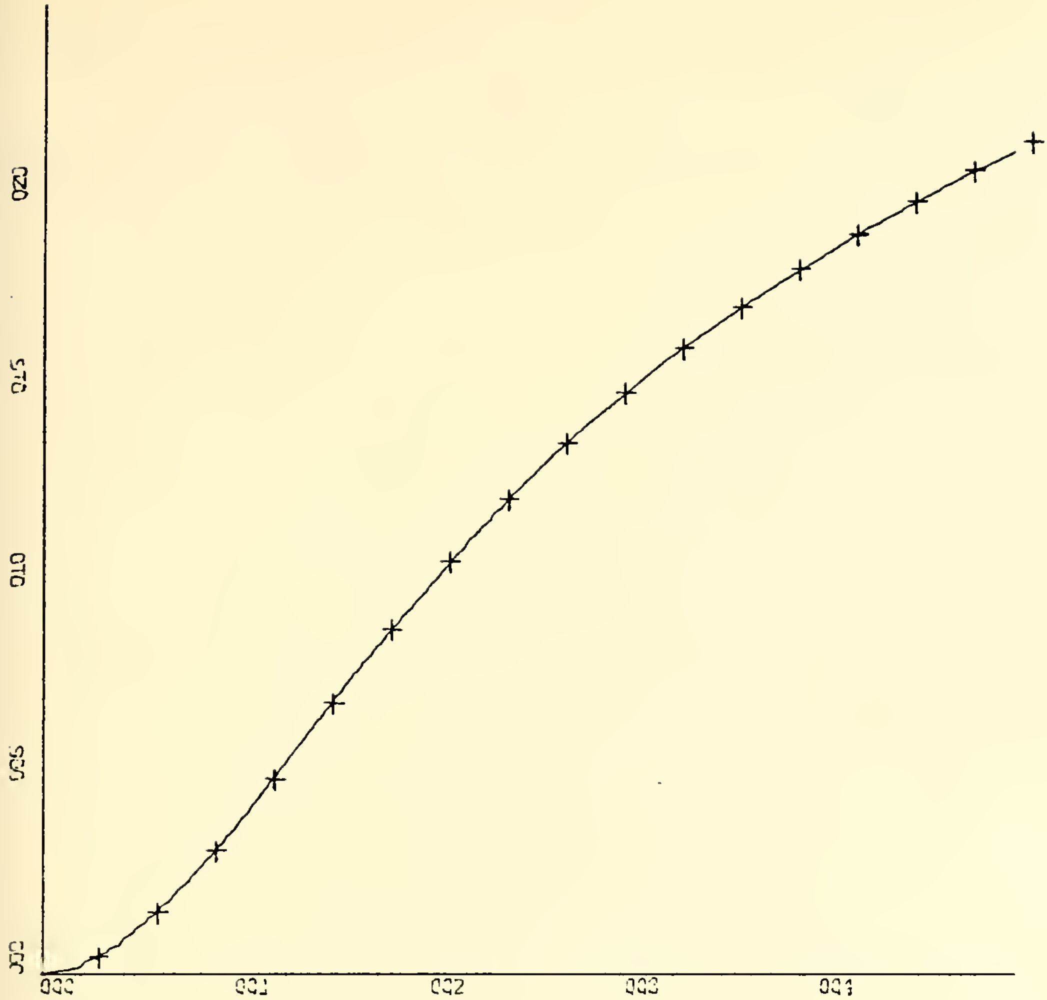
PROGRAM 4



x-scale = 1.0 units/inch
y-scale = 0.5 units/inch

FOR DELTA T = 0.2

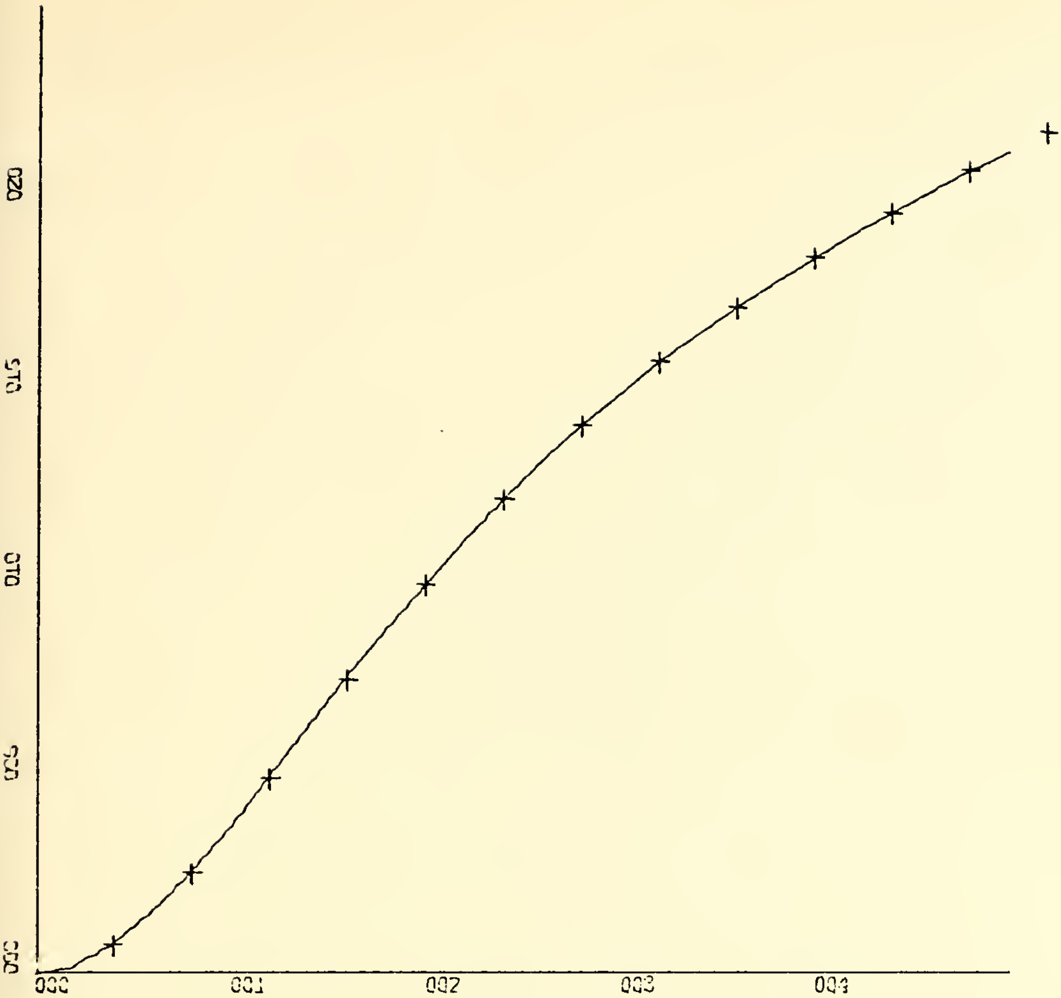
FIGURE 49



FOR DELTA T = 0.3 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.3 | 0.04158 |
| 0.6 | 0.15514 |
| 0.9 | 0.31700 |
| 1.2 | 0.50407 |
| 1.5 | 0.69801 |
| 1.8 | 0.88639 |
| 2.1 | 1.06213 |
| 2.4 | 1.22223 |
| 2.7 | 1.36641 |
| 3.0 | 1.49595 |
| 3.3 | 1.61280 |
| 3.6 | 1.71905 |
| 3.9 | 1.81664 |
| 4.2 | 1.90718 |
| 4.5 | 1.99200 |
| 4.8 | 2.07212 |
| 5.1 | 2.14834 |

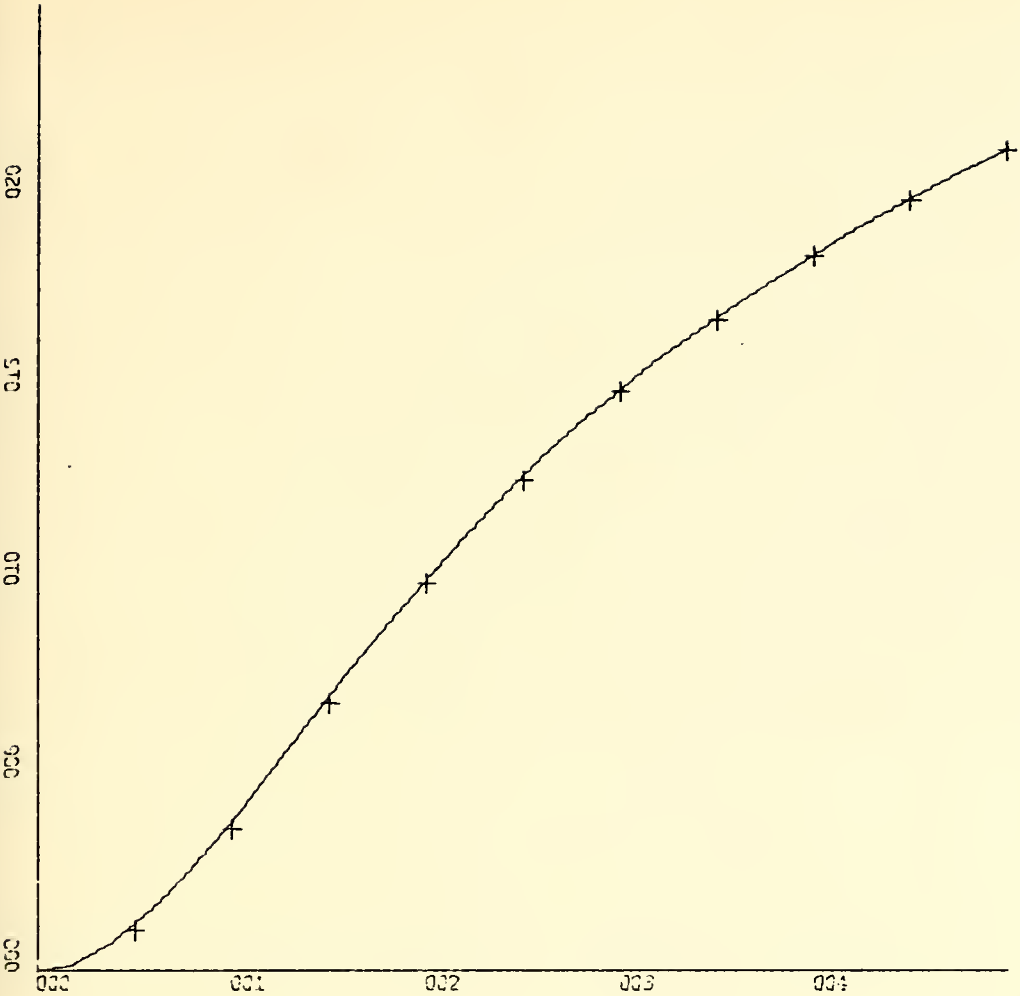
FIGURE 50



FOR DELTA T = 0.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.4 | 0.07102 |
| 0.8 | 0.25620 |
| 1.2 | 0.50068 |
| 1.6 | 0.75898 |
| 2.0 | 1.00322 |
| 2.4 | 1.22111 |
| 2.8 | 1.41066 |
| 3.2 | 1.57505 |
| 3.6 | 1.71916 |
| 4.0 | 1.84769 |
| 4.4 | 1.96446 |
| 4.8 | 2.07227 |
| 5.2 | 2.17312 |

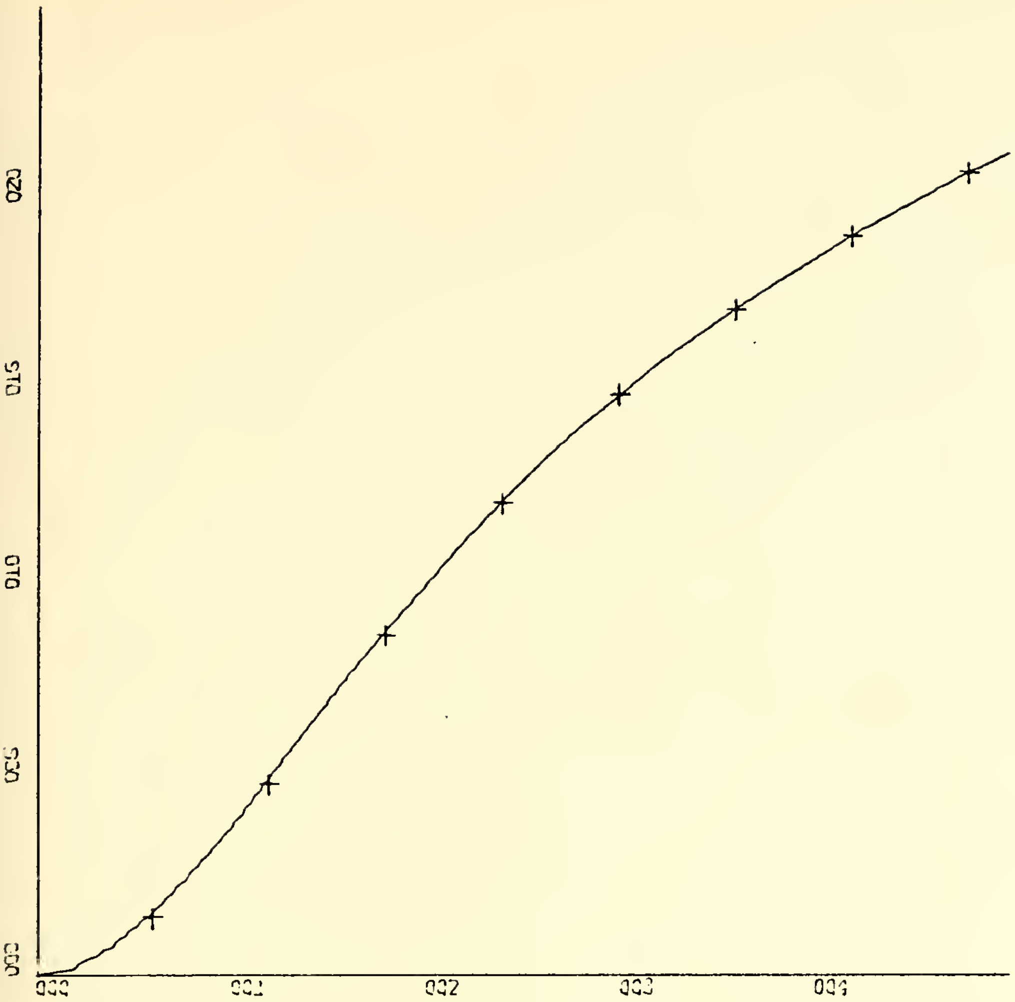
FIGURE 51



FOR DELTA T = 0.5 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.5 | 0.10622 |
| 1.0 | 0.36996 |
| 1.5 | 0.69092 |
| 2.0 | 1.00060 |
| 2.5 | 1.26987 |
| 3.0 | 1.49539 |
| 3.5 | 1.68484 |
| 4.0 | 1.84788 |
| 4.5 | 1.99233 |
| 5.0 | 2.12360 |

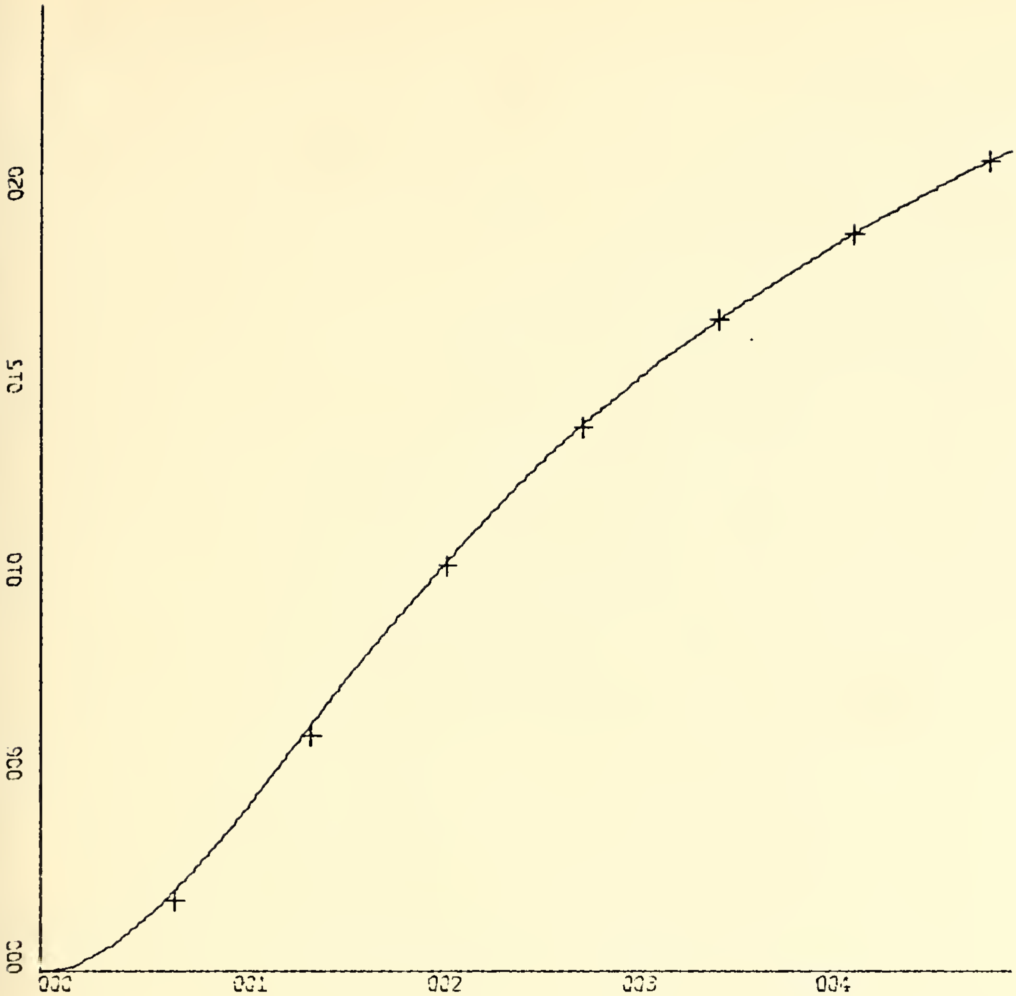
FIGURE 52



FOR DELTA T = 0.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.6 | 0.14606 |
| 1.2 | 0.49082 |
| 1.8 | 0.87667 |
| 2.4 | 1.21782 |
| 3.0 | 1.49500 |
| 3.6 | 1.71944 |
| 4.2 | 1.90777 |
| 4.8 | 2.07258 |

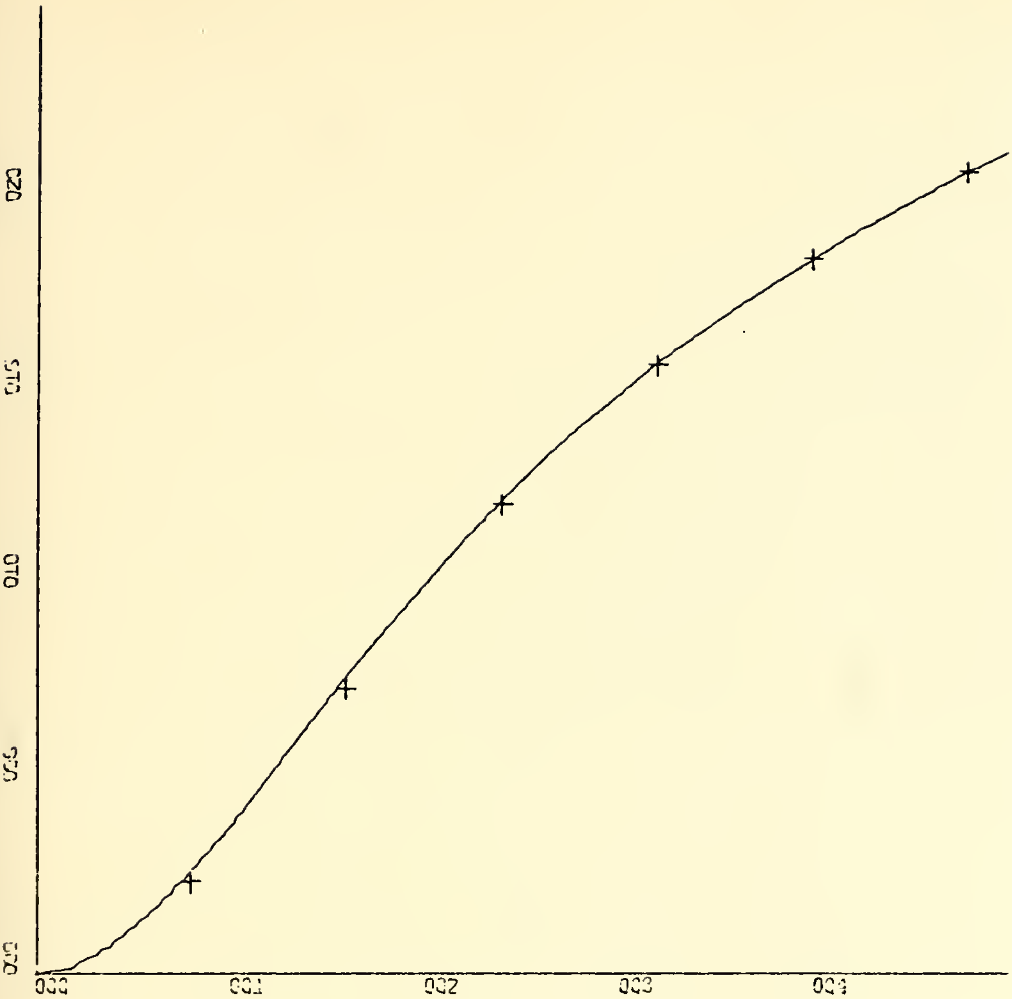
FIGURE 53



FOR DELTA T = 0.7 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.7 | 0.18951 |
| 1.4 | 0.61452 |
| 2.1 | 1.05164 |
| 2.8 | 1.40840 |
| 3.5 | 1.68514 |
| 4.2 | 1.90808 |
| 4.9 | 2.09855 |

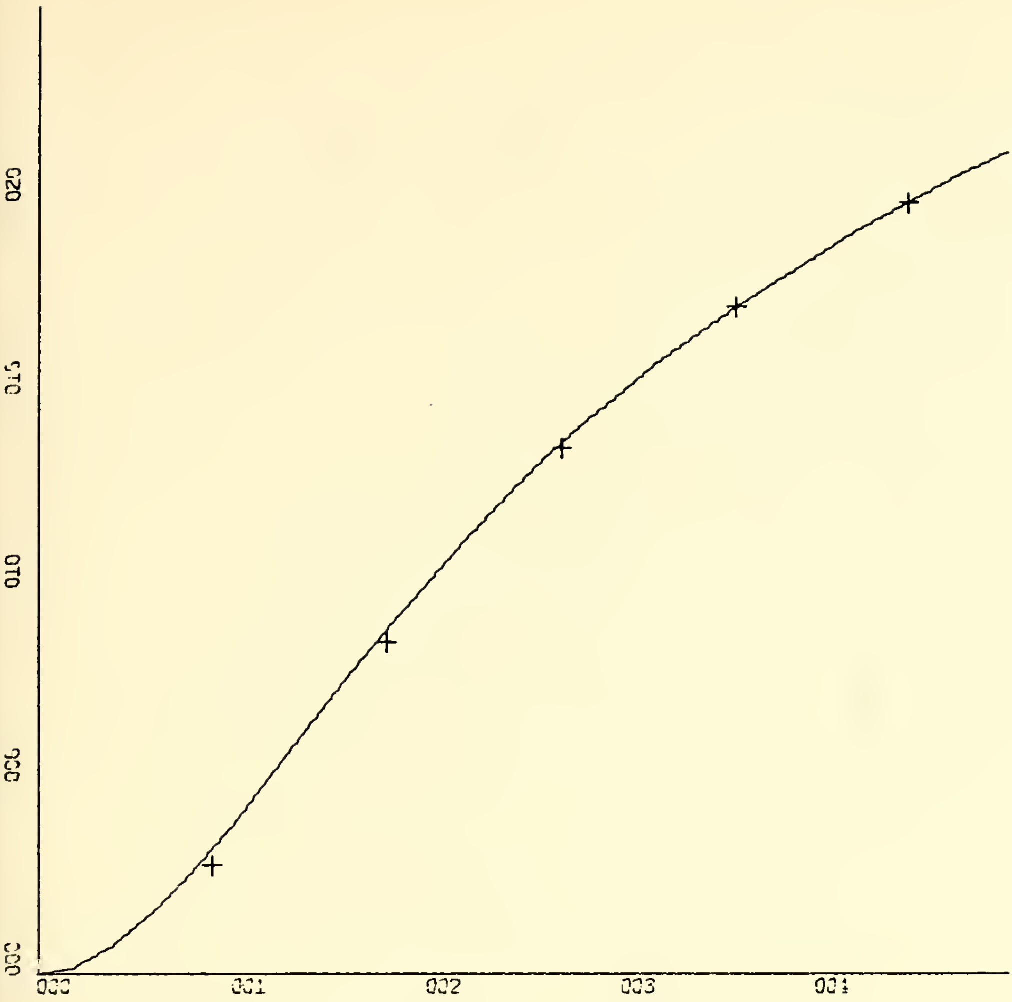
FIGURE 54



FOR DELTA T = 0.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.8 | 0.23568 |
| 1.6 | 0.73797 |
| 2.4 | 1.21295 |
| 3.2 | 1.57461 |
| 4.0 | 1.84884 |
| 4.8 | 2.07304 |

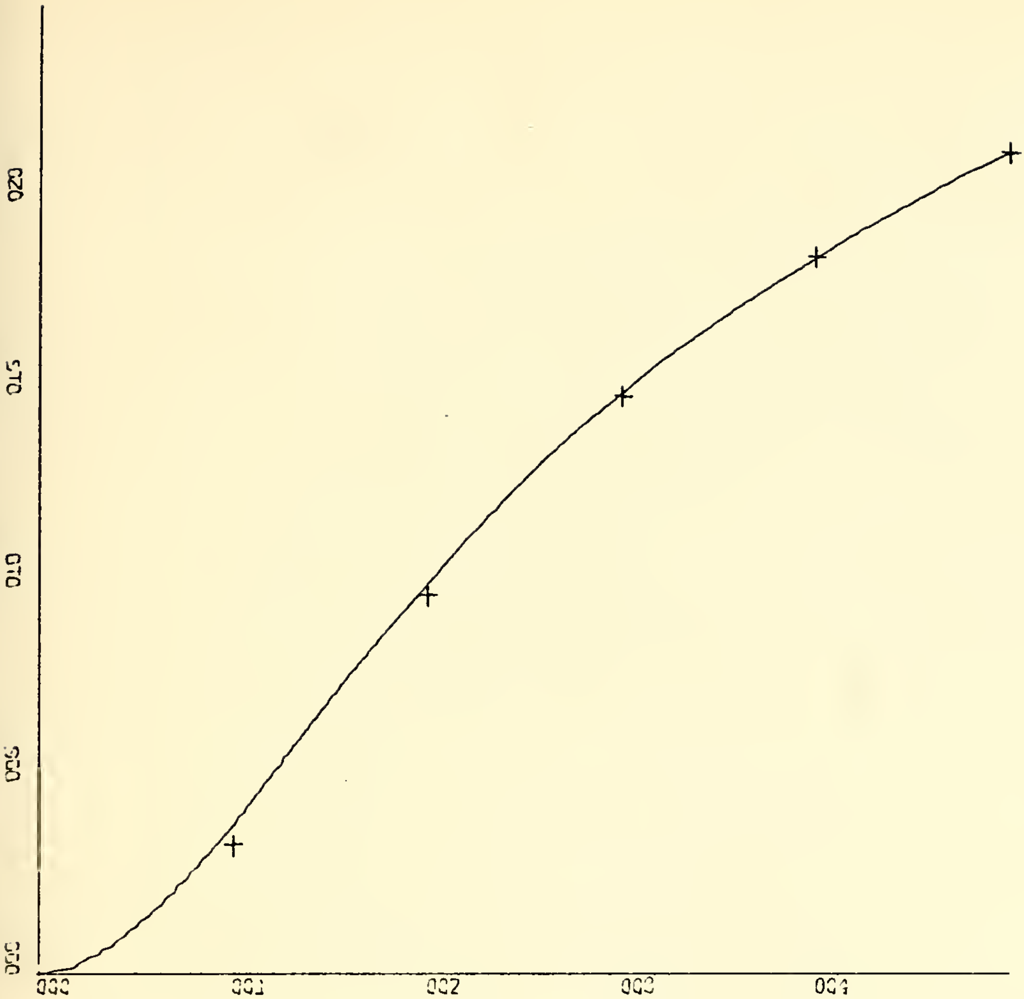
FIGURE 55



FOR DELTA T = 0.9 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.9 | 0.28383 |
| 1.8 | 0.85903 |
| 2.7 | 1.35987 |
| 3.6 | 1.72042 |
| 4.5 | 1.99350 |

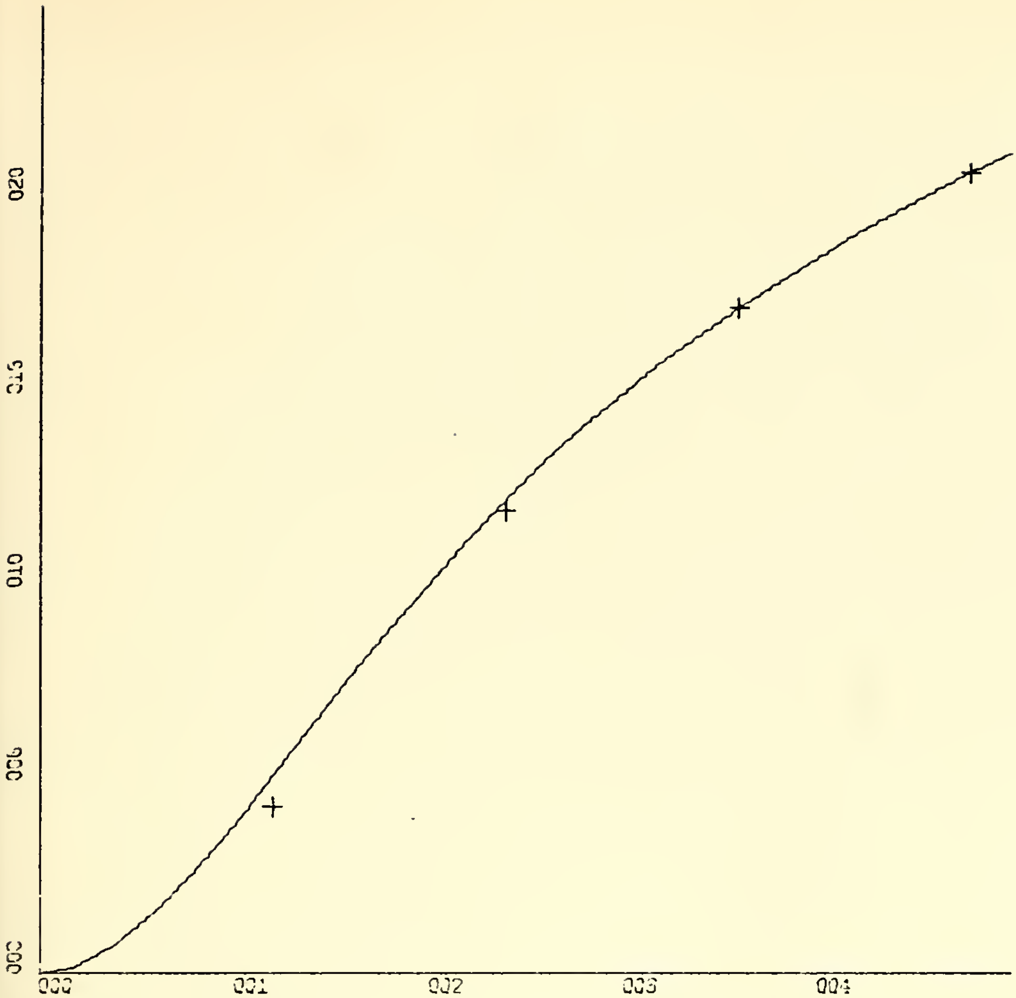
FIGURE 56



FOR DELTA T = 1.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.0 | 0.33333 |
| 2.0 | 0.97631 |
| 3.0 | 1.49299 |
| 4.0 | 1.85001 |
| 5.0 | 2.12461 |

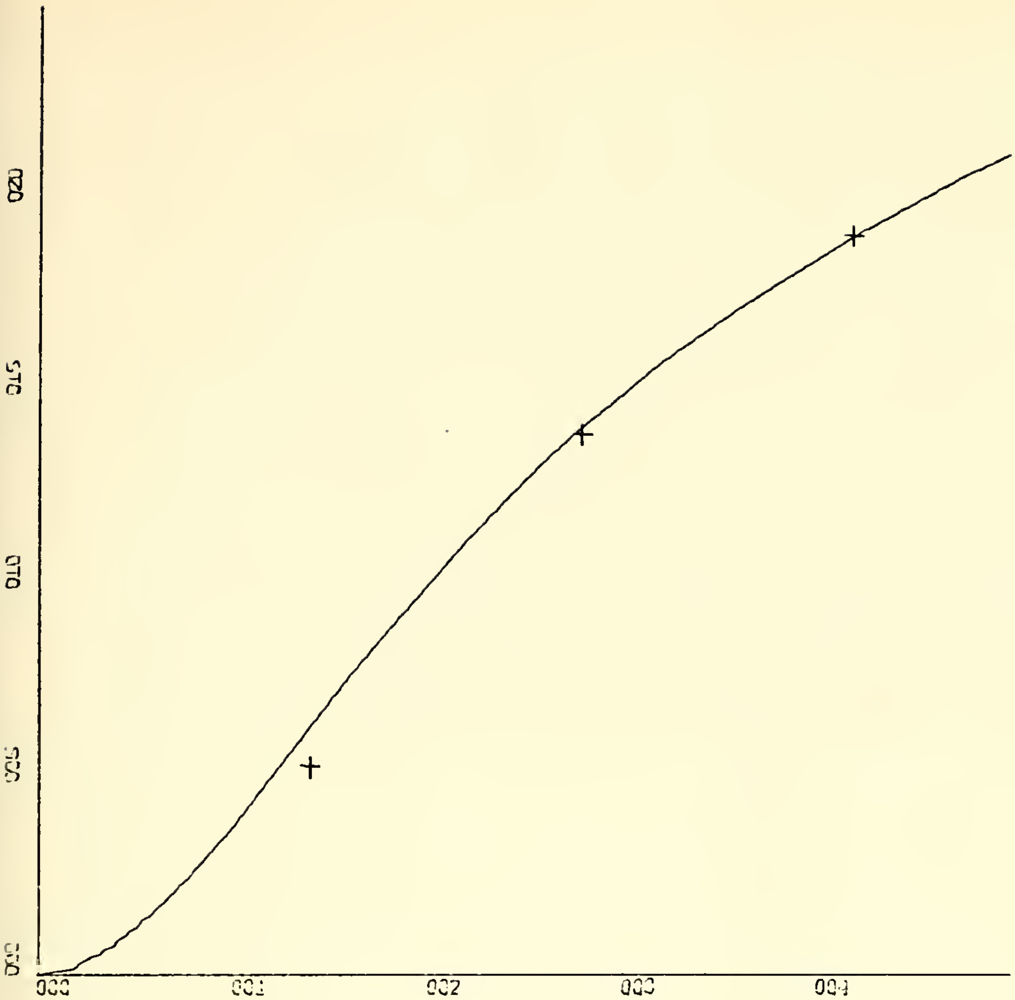
FIGURE 57



FOR DELTA T = 1.2 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.2 | 0.43445 |
| 2.4 | 1.19662 |
| 3.6 | 1.72293 |
| 4.8 | 2.07450 |

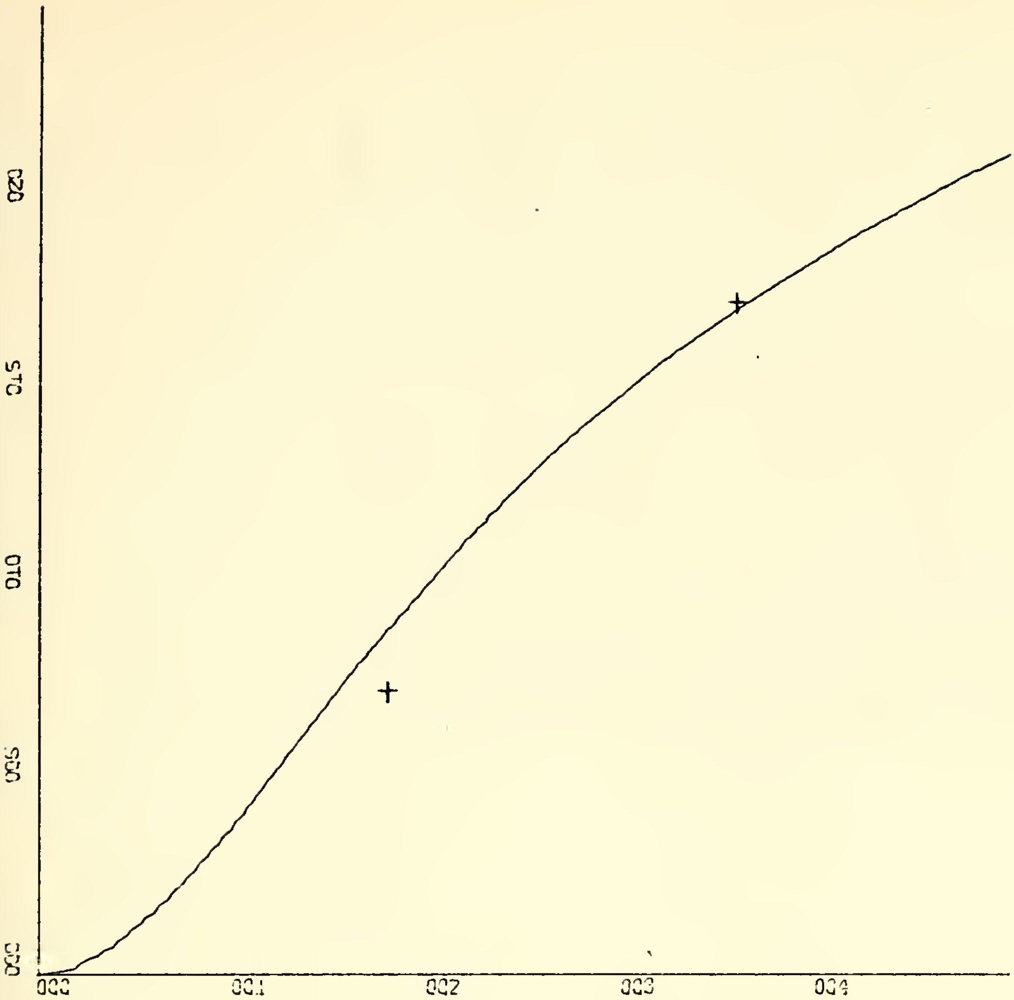
FIGURE 58



FOR DELTA T = 1.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.4 | 0.53603 |
| 2.8 | 1.39641 |
| 4.2 | 1.91440 |

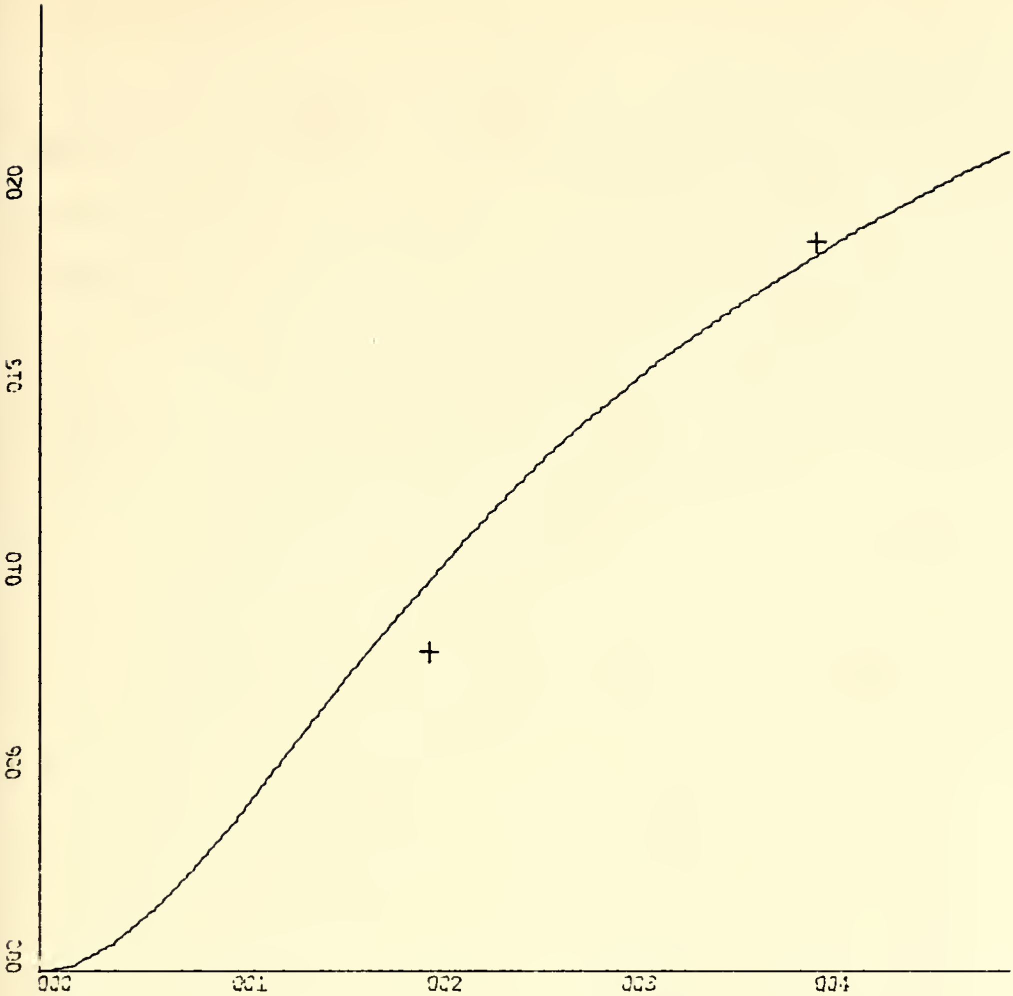
FIGURE 59



FOR DELTA T = 1.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.8 | 0.73387 |
| 3.6 | 1.73869 |

FIGURE 60




FOR DELTA T = 2.    TFIN = 5.2

| T | Q(N) |
|---|---|
| 2.0 | 0.82843 |

FIGURE 61

1. Case 5 – $\dot{Y} + tY = t$

As case 2 follows case 1, as calculations are concerned, case 5 follows case 4 with the only difference that P(t) is now equal to t and this must be the value used to substitute in P before and during the division steps.

Graphs, plots and solution values for several time iterations follow (Figs. 62-75). Computer algorithm for solution of this problem is shown in Program 5 (page 95).

FOR DELTA T = 0.2 TFIN = 5.2

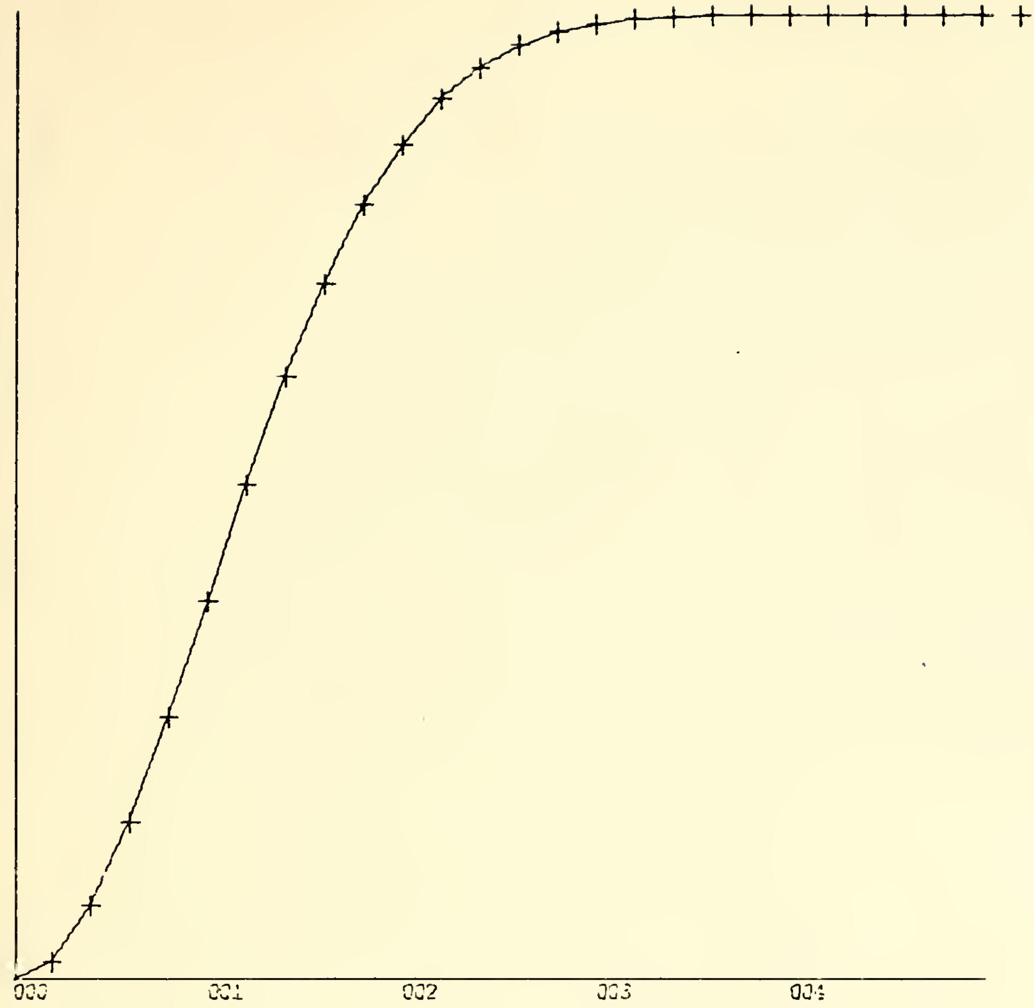
| T | Q(N) | PR |
|---|---|---|
| 0.0 | 0.0 | 0.0 |
| 0.2 | 0.01961 | 0.01980 |
| 0.4 | 0.07617 | 0.07688 |
| 0.6 | 0.16332 | 0.16473 |
| 0.8 | 0.27178 | 0.27385 |
| 1.0 | 0.39094 | 0.39347 |
| 1.2 | 0.51058 | 0.51324 |
| 1.4 | 0.62220 | 0.62468 |
| 1.6 | 0.71991 | 0.72186 |
| 1.8 | 0.80061 | 0.80209 |
| 2.0 | 0.86375 | 0.86466 |
| 2.2 | 0.91066 | 0.91107 |
| 2.4 | 0.94380 | 0.94386 |
| 2.6 | 0.96617 | 0.96595 |
| 2.8 | 0.98040 | 0.98016 |
| 3.0 | 0.98914 | 0.98889 |
| 3.2 | 0.99424 | 0.99402 |
| 3.4 | 0.99707 | 0.99691 |
| 3.6 | 0.99857 | 0.99847 |
| 3.8 | 0.99933 | 0.99927 |
| 4.0 | 0.99970 | 0.99966 |
| 4.2 | 0.99986 | 0.99985 |
| 4.4 | 0.99993 | 0.99994 |
| 4.6 | 0.99996 | 0.99997 |
| 4.8 | 0.99997 | 0.99999 |
| 5.0 | 0.99997 | 1.00000 |

Equation (46) is repeated here for convenience:

$$Y_A^*(z) = \frac{T^2 z^{-1} + T^2 z^{-2}}{(2+PT) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC3C
C
C   Q(T)=RAMP   &   P(T)=T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = T
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('   XX=',7X,'    PR=')
      END
```

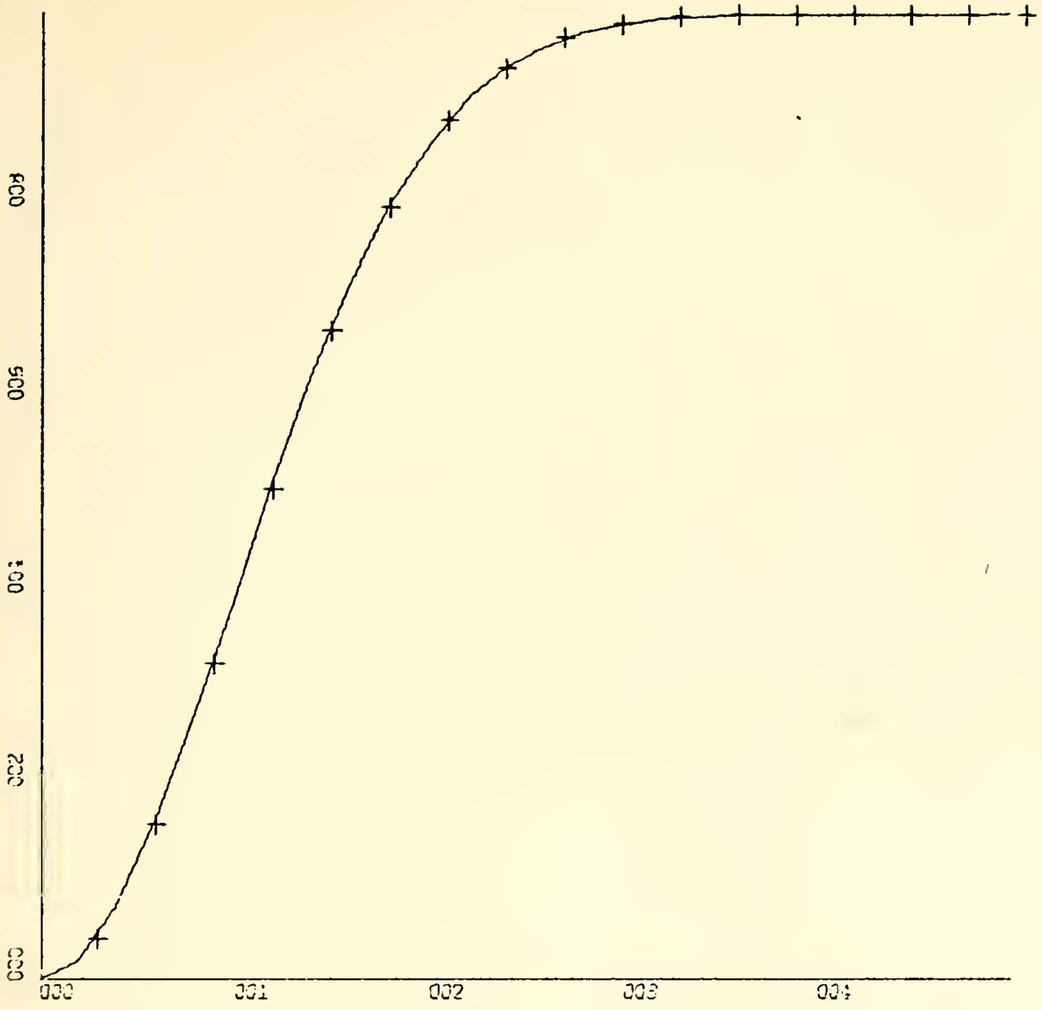
PROGRAM 5



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

DELTA T = 0.2

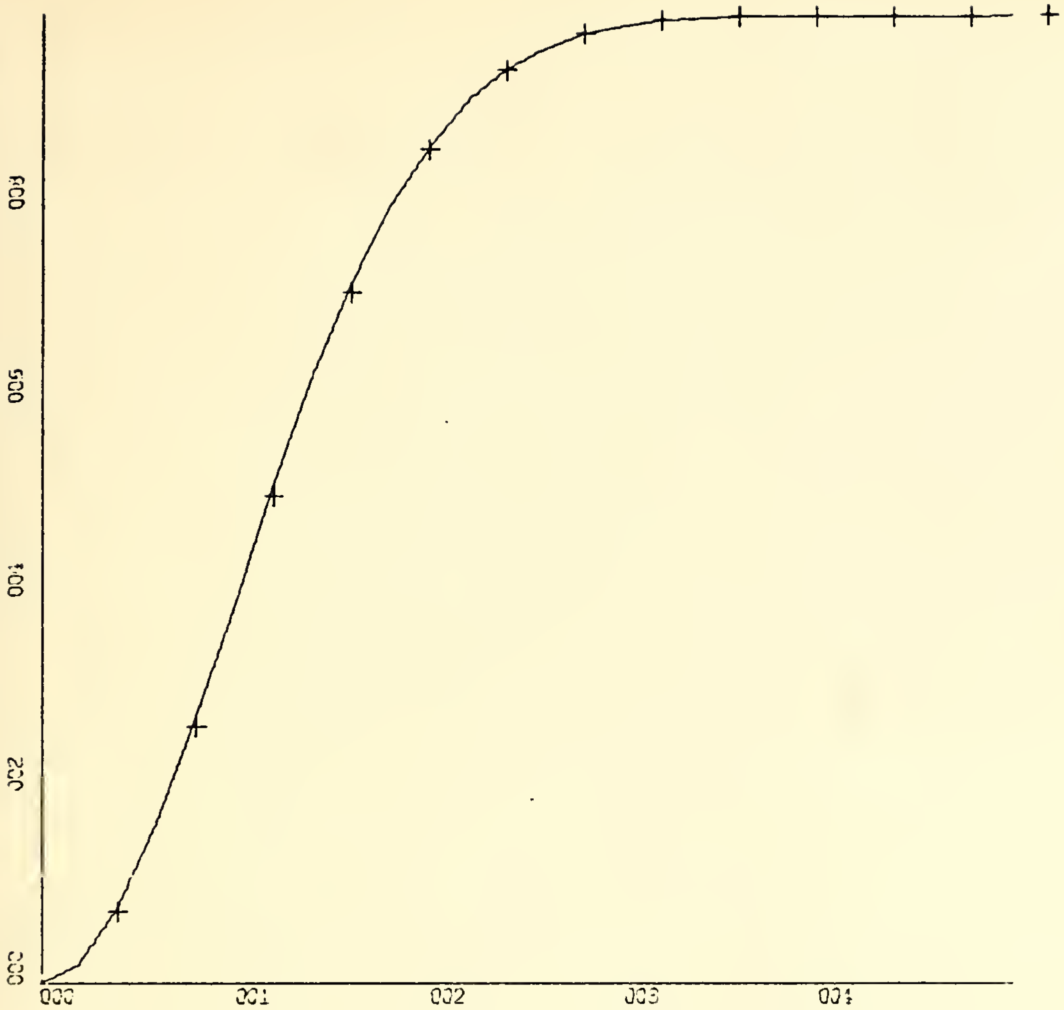
FIGURE 62



FOR DELTA T = 0.3    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.3 | 0.04306 |
| 0.6 | 0.16158 |
| 0.9 | 0.32779 |
| 1.2 | 0.50723 |
| 1.5 | 0.67015 |
| 1.8 | 0.79871 |
| 2.1 | 0.88826 |
| 2.4 | 0.94372 |
| 2.7 | 0.97436 |
| 3.0 | 0.98948 |
| 3.3 | 0.99613 |
| 3.6 | 0.99873 |
| 3.9 | 0.99963 |
| 4.2 | 0.99990 |
| 4.5 | 0.99997 |
| 4.8 | 0.99999 |
| 5.1 | 0.99999 |

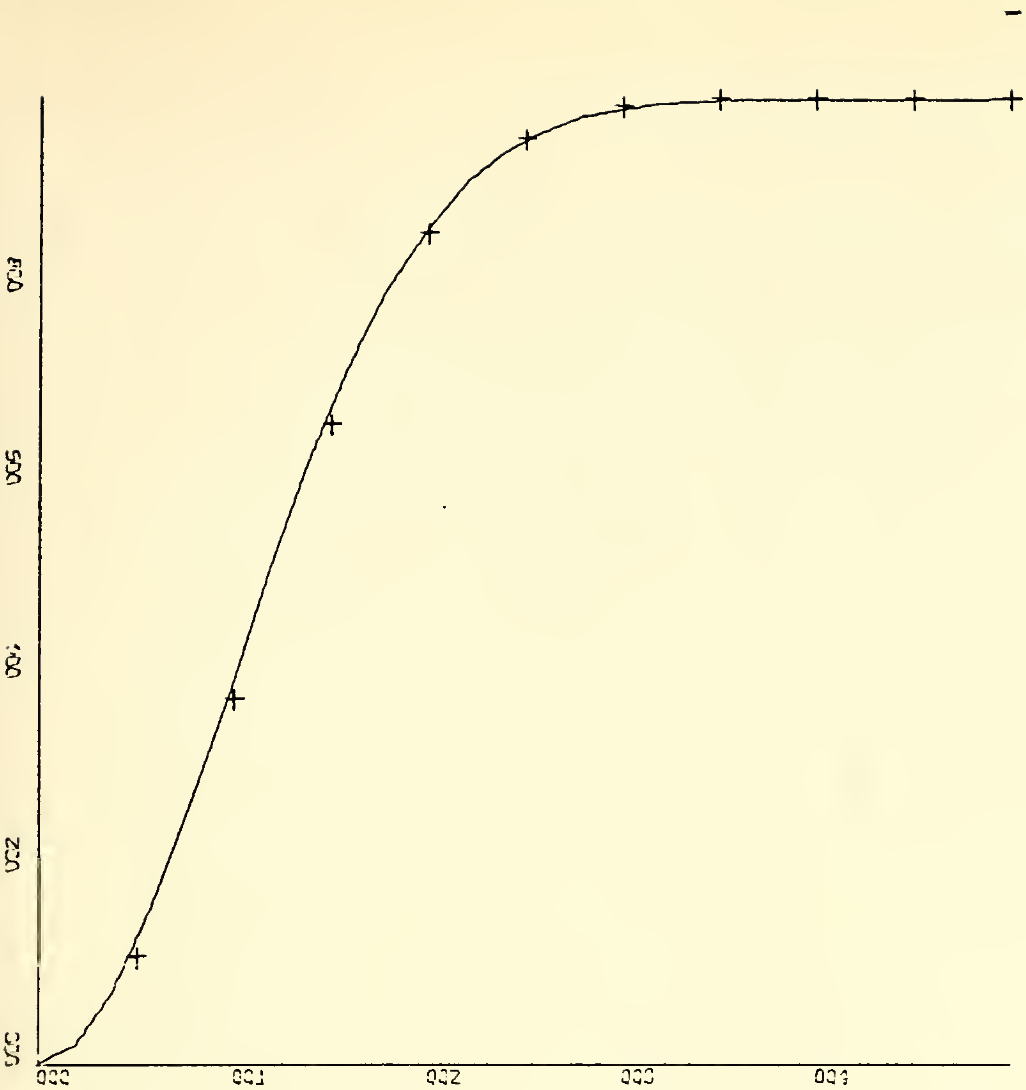
FIGURE 63



FOR DELTA T = 0.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.4 | 0.07407 |
| 0.8 | 0.26564 |
| 1.2 | 0.50253 |
| 1.6 | 0.71358 |
| 2.0 | 0.86088 |
| 2.4 | 0.94360 |
| 2.8 | 0.98120 |
| 3.2 | 0.99496 |
| 3.6 | 0.99895 |
| 4.0 | 0.99984 |
| 4.4 | 0.99998 |
| 4.8 | 1.00000 |
| 5.2 | 1.00000 |

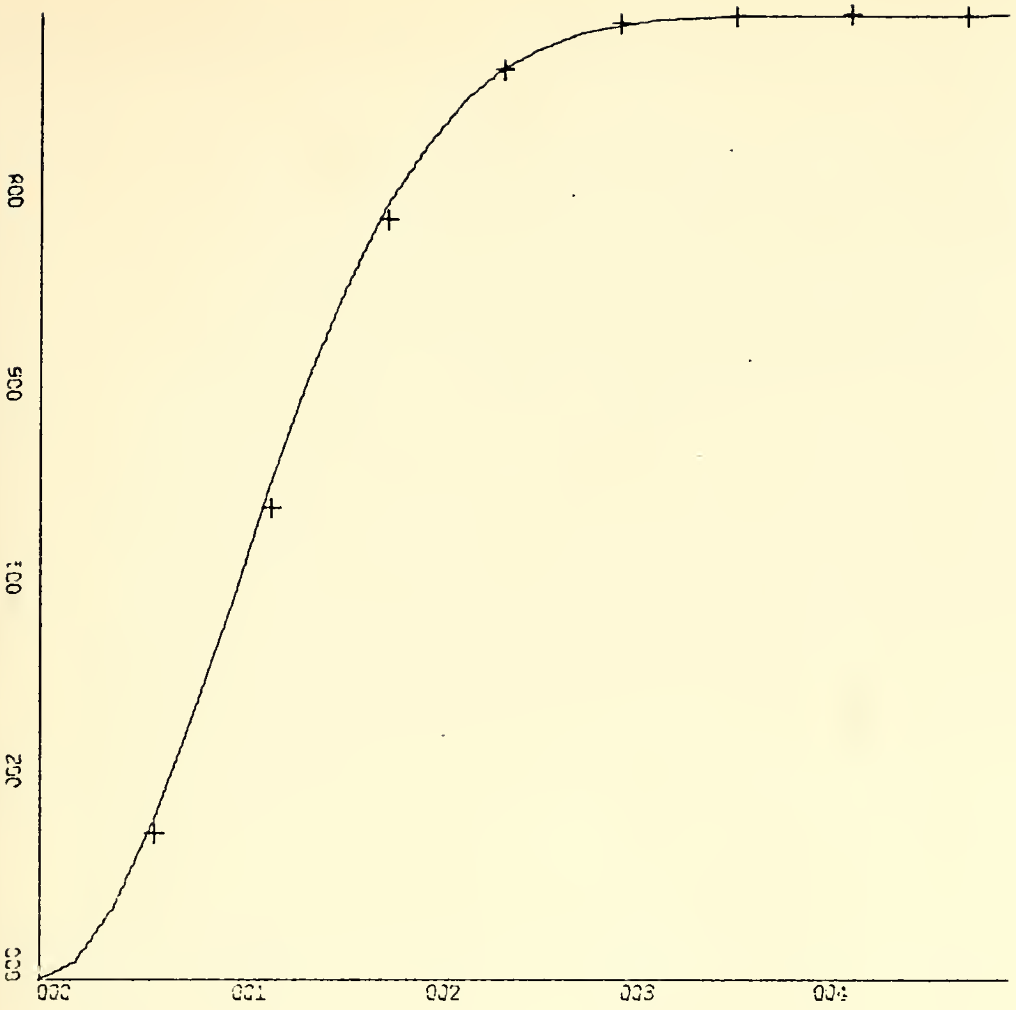
FIGURE 64



FOR DELTA T = 0.5 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.5 | 0.11111 |
| 1.0 | 0.37776 |
| 1.5 | 0.66061 |
| 2.0 | 0.85859 |
| 2.5 | 0.95649 |
| 3.0 | 0.99067 |
| 3.5 | 0.99876 |
| 4.0 | 0.99992 |
| 4.5 | 1.00000 |
| 5.0 | 1.00000 |

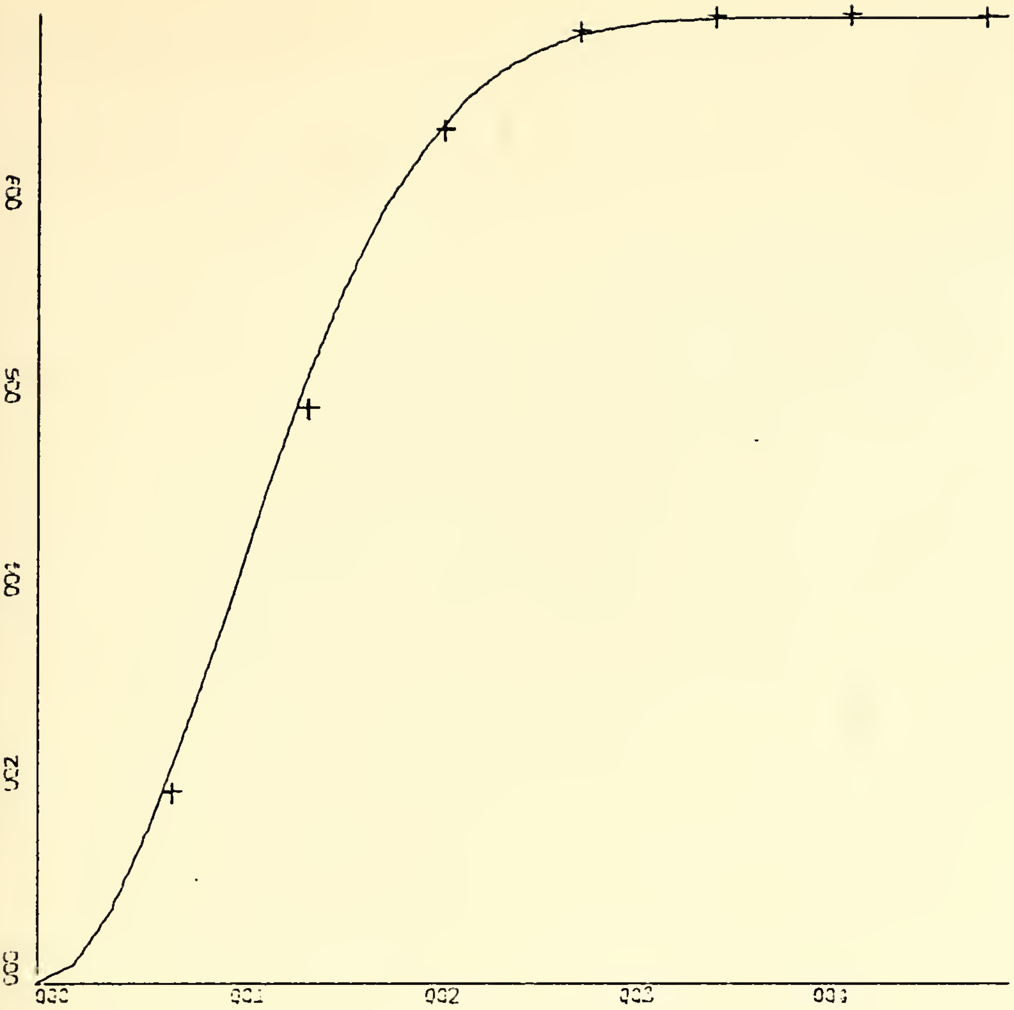
FIGURE 65



FOR DELTA T = 0.6    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.6 | 0.15254 |
| 1.2 | 0.48903 |
| 1.8 | 0.78765 |
| 2.4 | 0.94321 |
| 3.0 | 0.99163 |
| 3.6 | 0.99960 |
| 4.2 | 1.00001 |
| 4.8 | 1.00000 |

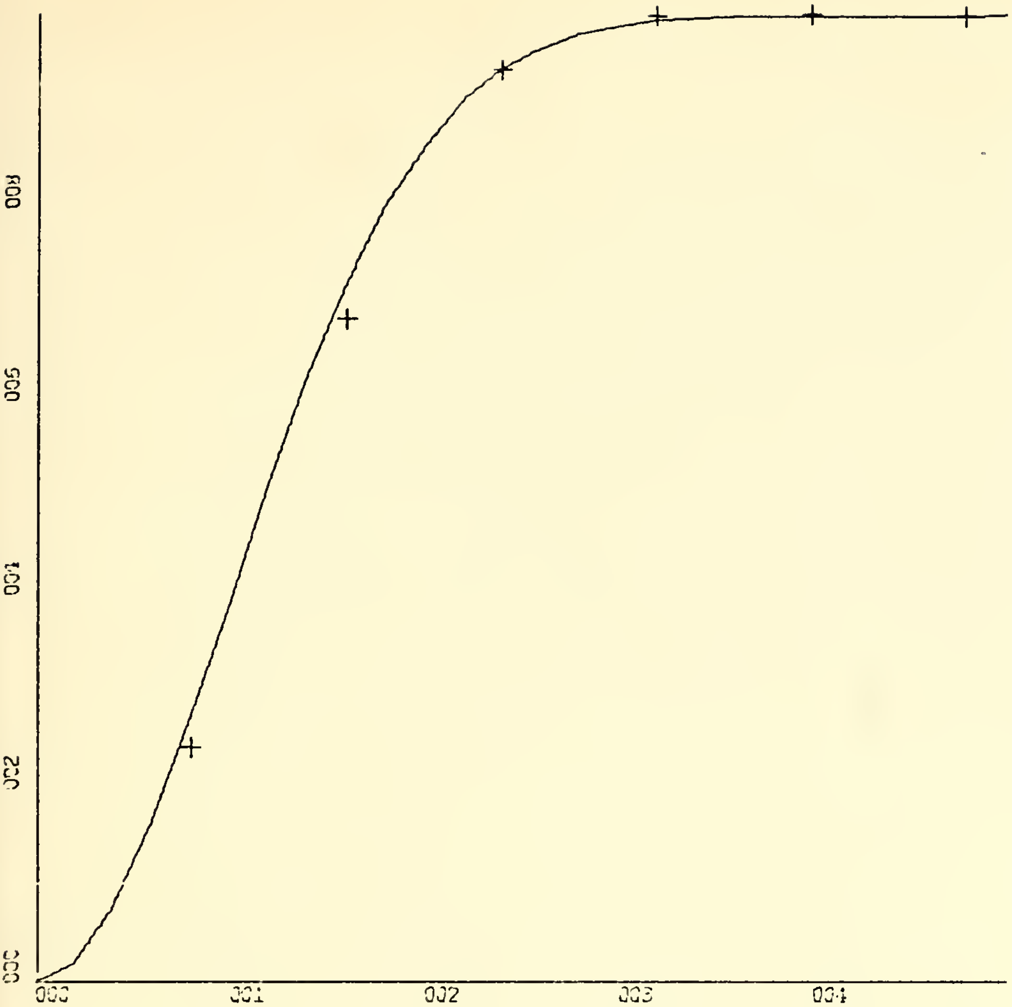
FIGURE 66



FOR DELTA T = 0.7 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.7 | 0.19679 |
| 1.4 | 0.59300 |
| 2.1 | 0.88036 |
| 2.8 | 0.98399 |
| 3.5 | 0.99986 |
| 4.2 | 1.00001 |
| 4.9 | 1.00000 |

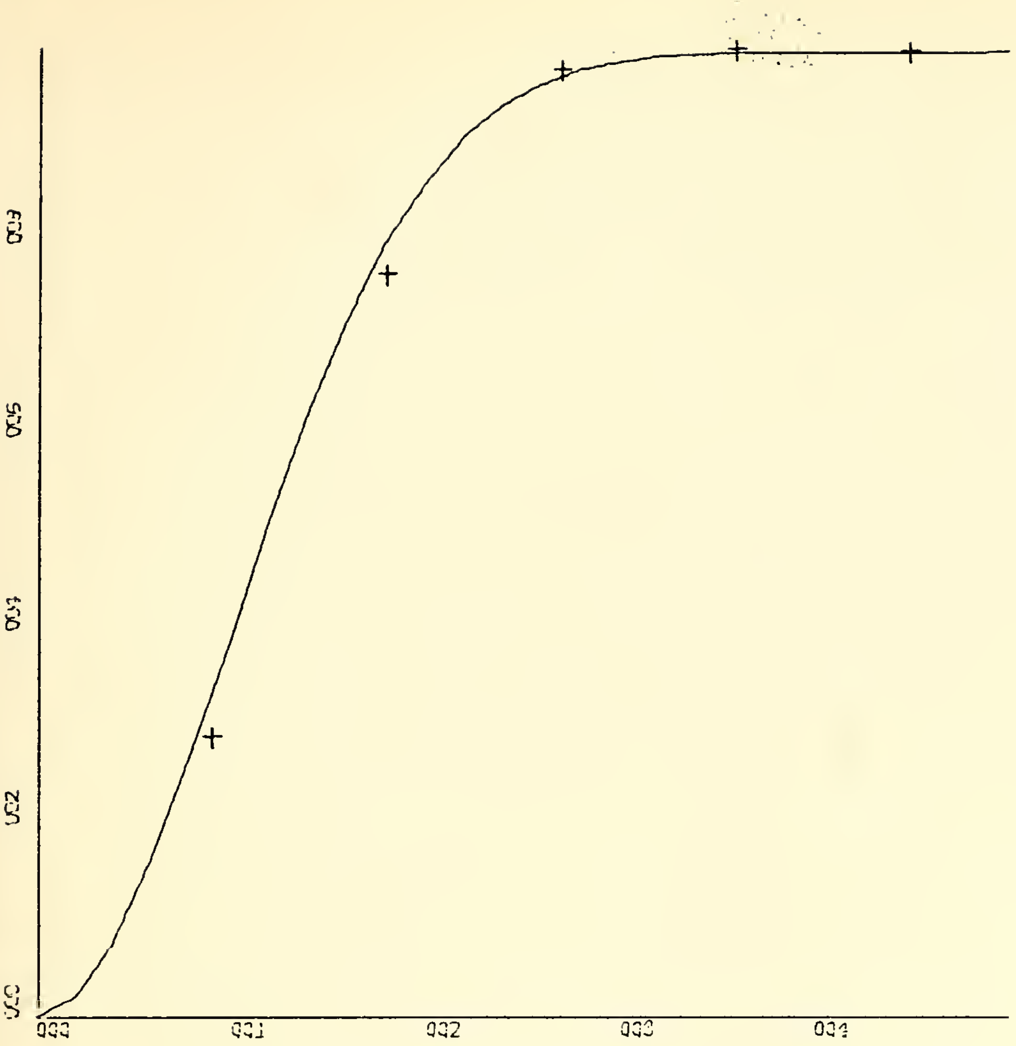
FIGURE 67



FOR DELTA T = 0.8    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.8 | 0.24242 |
| 1.6 | 0.68588 |
| 2.4 | 0.94231 |
| 3.2 | 0.99899 |
| 4.0 | 1.00011 |
| 4.8 | 0.99998 |

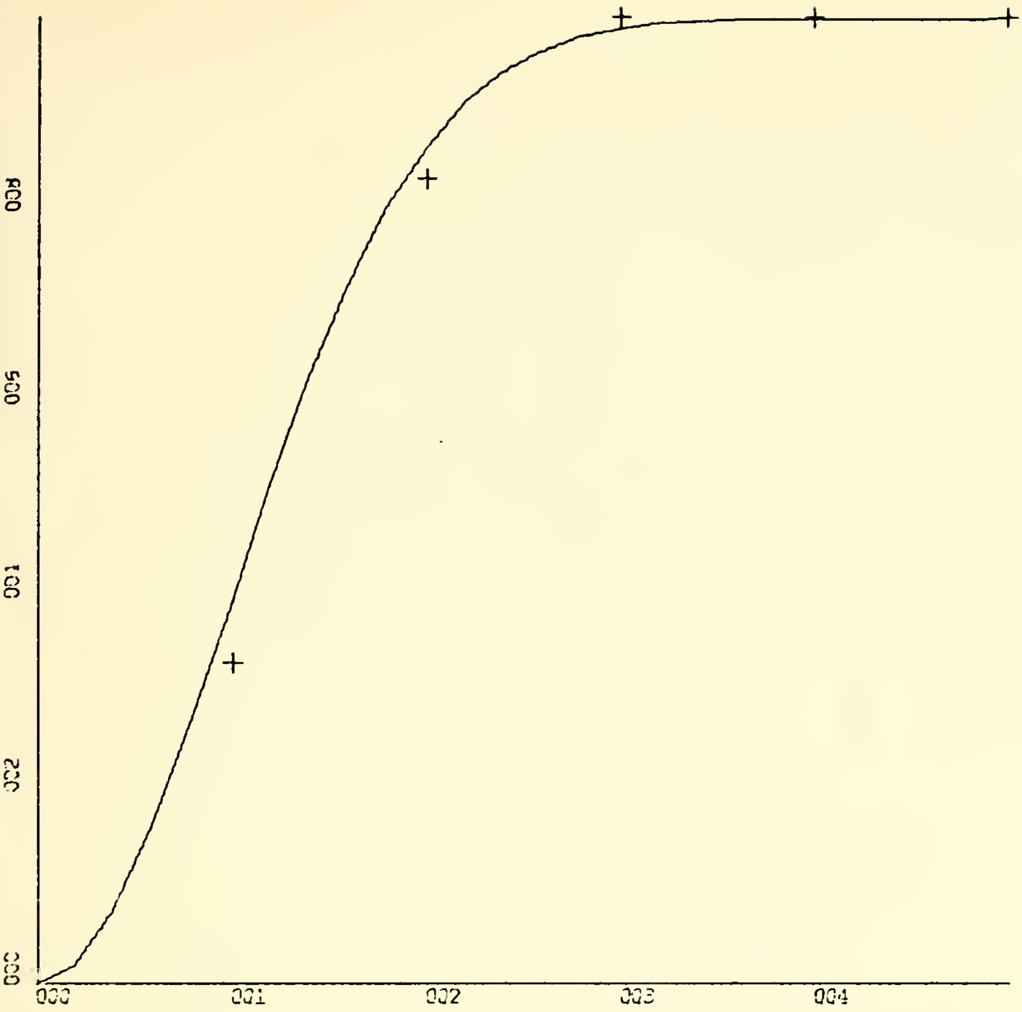
FIGURE 68



FOR DELTA T = 0.9    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.9 | 0.28826 |
| 1.8 | 0.76603 |
| 2.7 | 0.97993 |
| 3.6 | 1.00165 |
| 4.5 | 0.99966 |

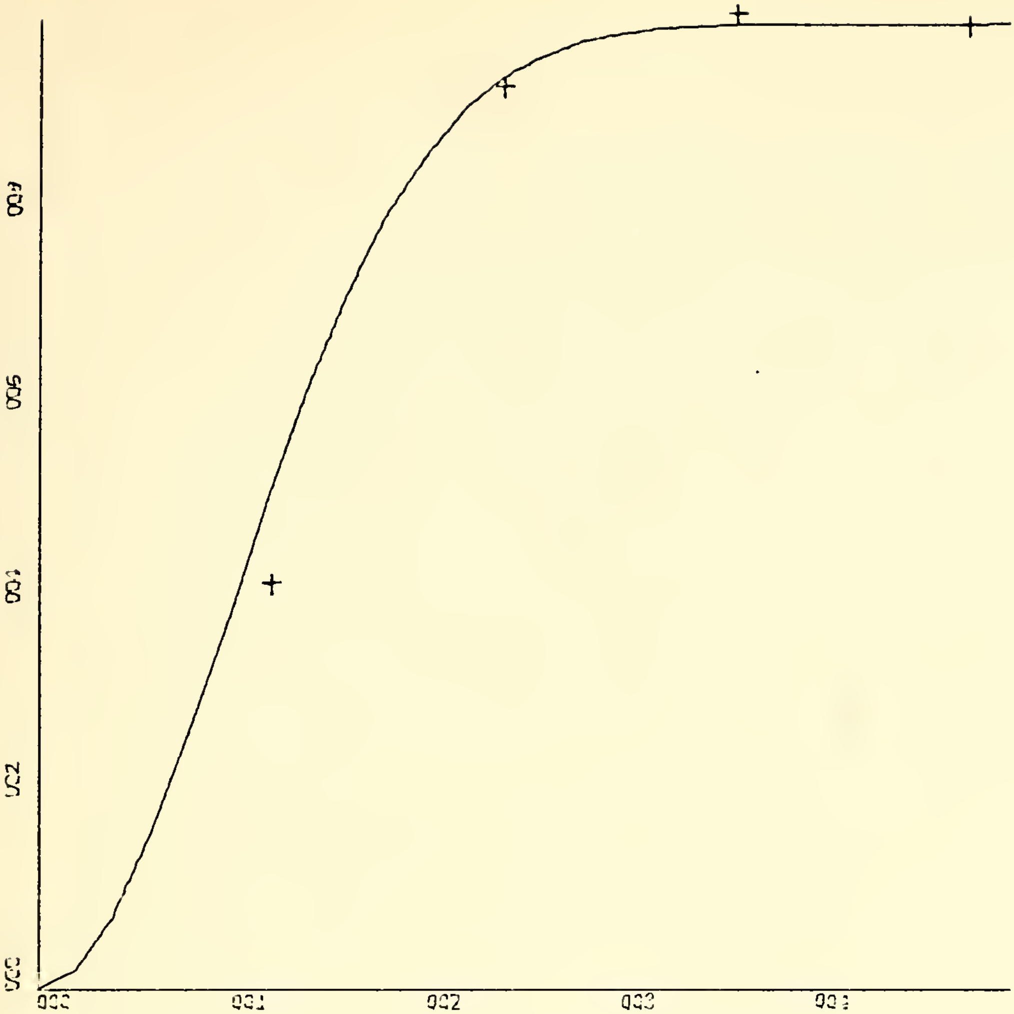
FIGURE 69




FOR DELTA T = 1.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.0 | 0.33333 |
| 2.0 | 0.83333 |
| 3.0 | 1.00000 |
| 4.0 | 1.00000 |
| 5.0 | 1.00000 |

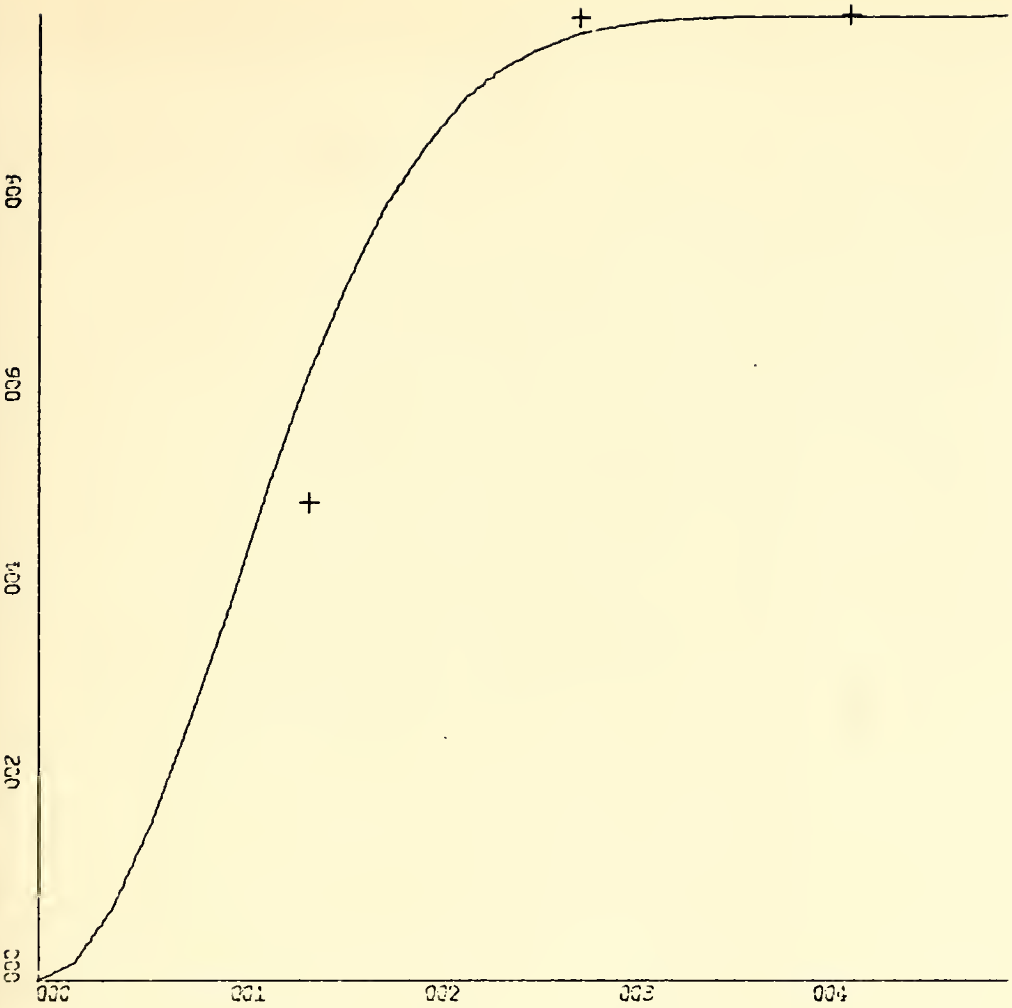
FIGURE 70



FOR DELTA T = 1.2    TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.2 | 0.41860 |
| 2.4 | 0.93328 |
| 3.6 | 1.00929 |
| 4.8 | 0.99722 |

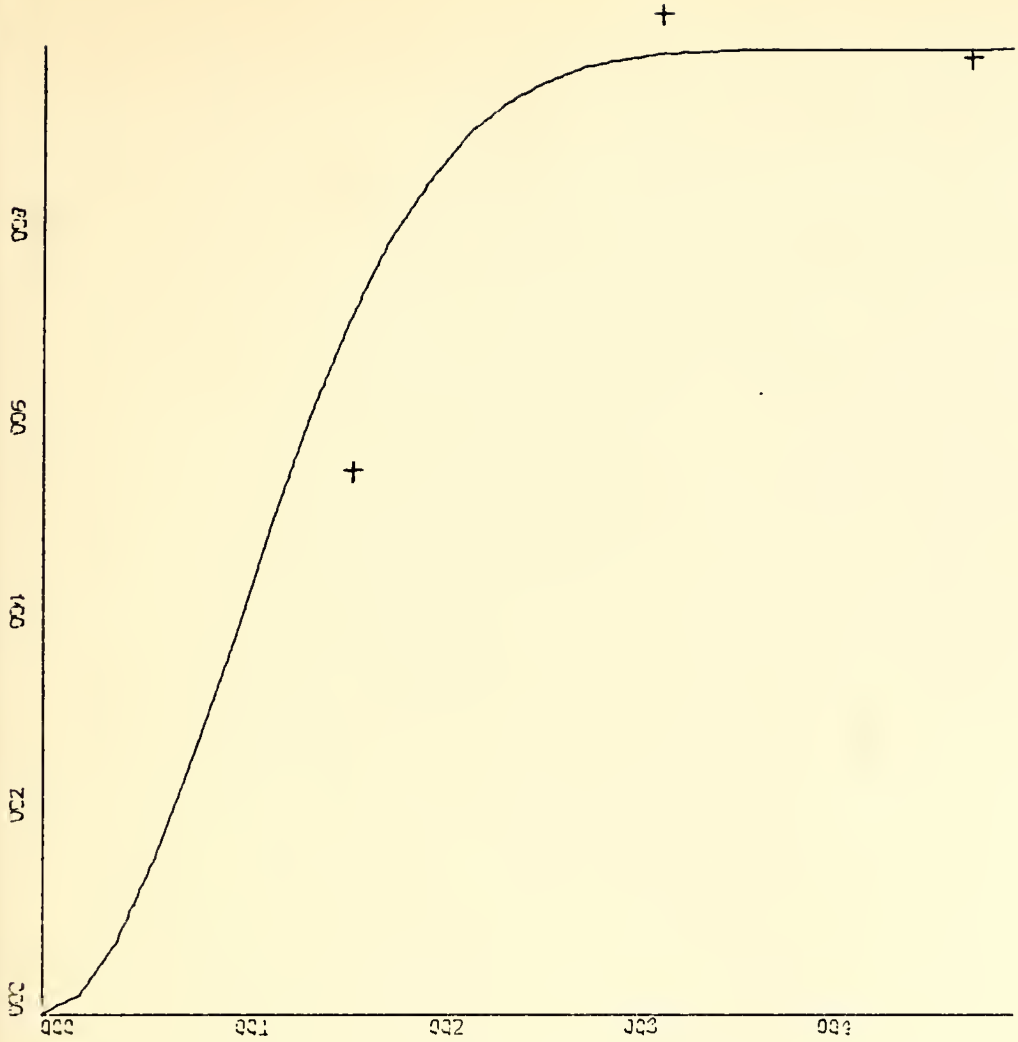
FIGURE 71



FOR DELTA T = 1.4   TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.4 | 0.49495 |
| 2.8 | 0.99659 |
| 4.2 | 1.00083 |

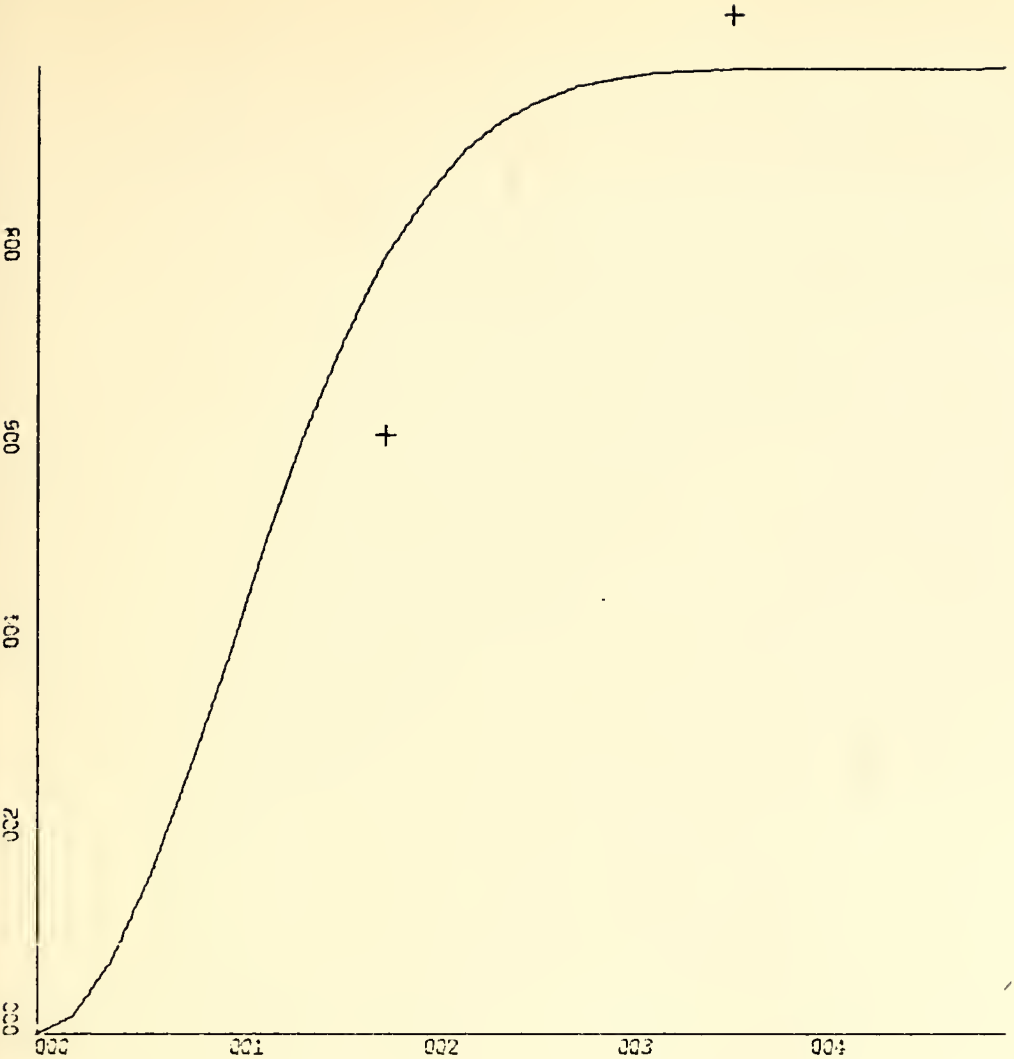
FIGURE 72



FOR DELTA T = 1.6    TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.6 | 0.56140 |
| 3.2 | 1.03450 |
| 4.8 | 0.98888 |

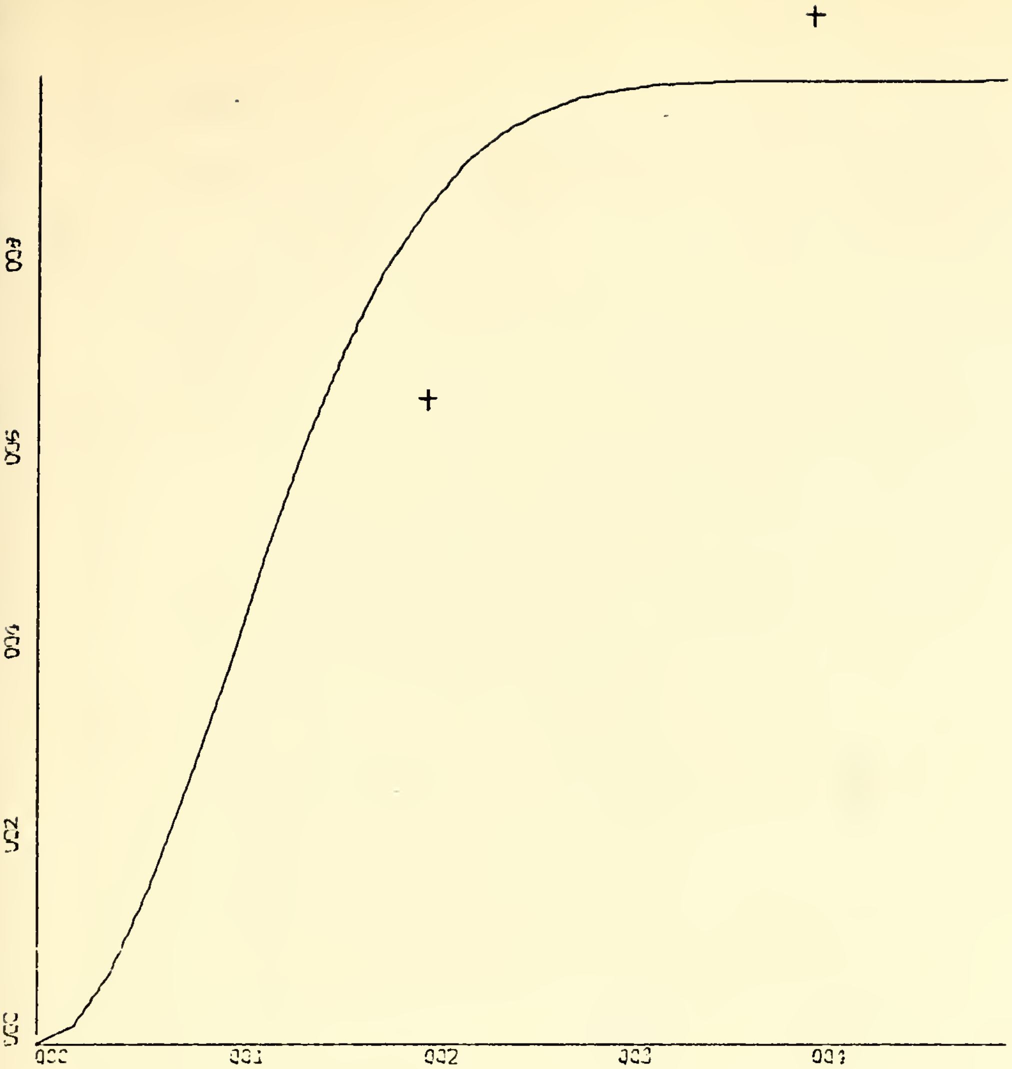
FIGURE 73




FOR DELTA T = 1.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.8 | 0.61832 |
| 3.6 | 1.05581 |

FIGURE 74



FOR DELTA T = 2.0    TFIN = 5.2

| T | Q(N) |
|---|---|
| 2.0 | 0.66667 |
| 4.0 | 1.06667 |

FIGURE 75

2. Case 6 — $\dot{Y} + t^2 Y = t$

Calculations follow exactly Case 4 except for $P(t) = t^2$. In the division steps, substitution of P for its value $t^2$ must be done in this case.

Solutions for several time iterations follow (Figs. 76 to 89). Computer solution is in Program 6 (page 111).

FOR DELTA T = 0.2 TFIN - 5.2

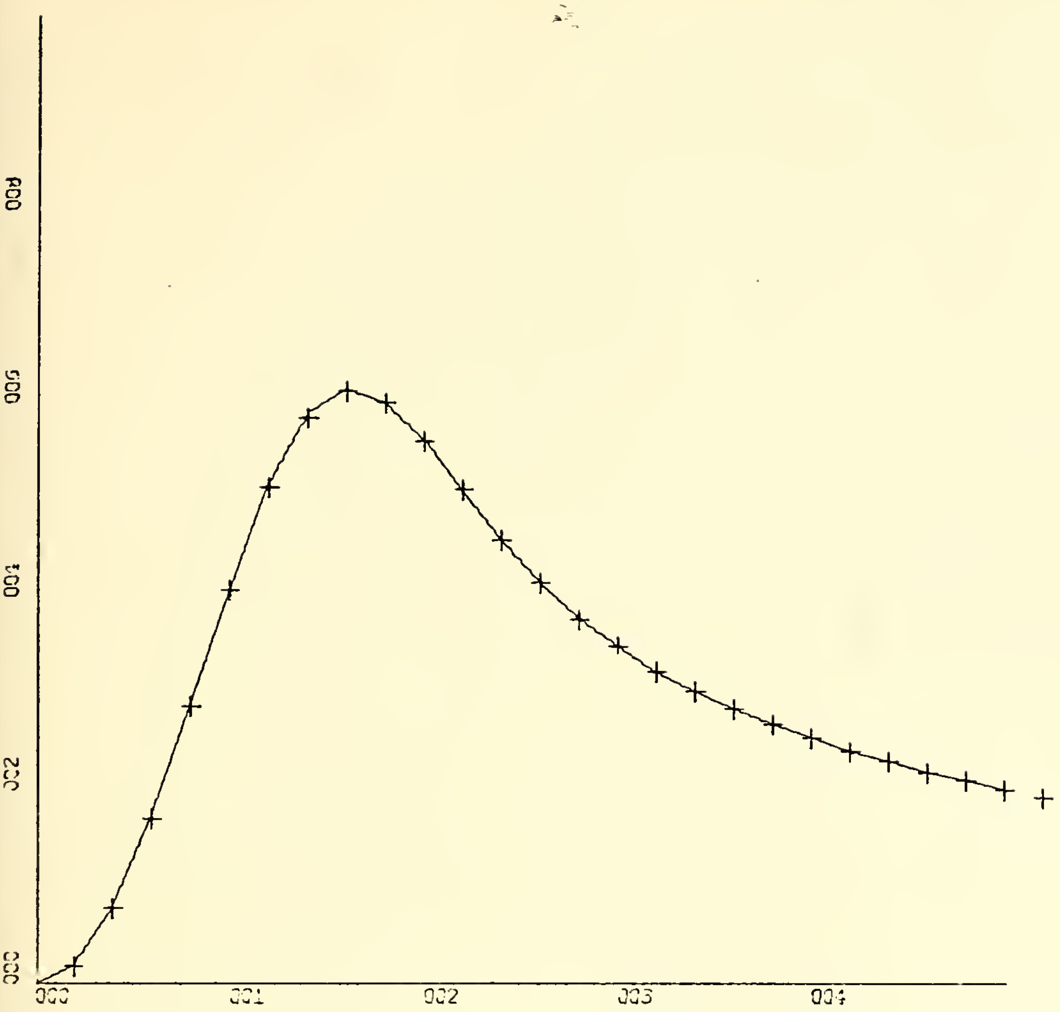
| T | Q(N) | PR |
|---|---|---|
| 0.2 | 0.01992 | 0.01997 |
| 0.4 | 0.07858 | 0.07898 |
| 0.6 | 0.17116 | 0.17243 |
| 0.8 | 0.28666 | 0.28923 |
| 1.0 | 0.40755 | 0.41144 |
| 1.2 | 0.51294 | 0.51742 |
| 1.4 | 0.58451 | 0.58837 |
| 1.6 | 0.61301 | 0.61523 |
| 1.8 | 0.60127 | 0.60163 |
| 2.0 | 0.56176 | 0.56095 |
| 2.2 | 0.51015 | 0.50914 |
| 2.4 | 0.45891 | 0.45829 |
| 2.6 | 0.41443 | 0.41425 |
| 2.8 | 0.37796 | 0.37801 |
| 3.0 | 0.34824 | 0.34833 |
| 3.2 | 0.32354 | 0.32360 |
| 3.4 | 0.30253 | 0.30257 |
| 3.6 | 0.28433 | 0.28436 |
| 3.8 | 0.26835 | 0.26837 |
| 4.0 | 0.25418 | 0.25420 |
| 4.2 | 0.24150 | 0.24152 |
| 4.4 | 0.23008 | 0.23009 |
| 4.6 | 0.21973 | 0.21974 |
| 4.8 | 0.21029 | 0.21030 |
| 5.0 | 0.20166 | 0.20167 |
| 5.2 | 0.19372 | |

Equation (46) is repeated here for convenience:

$$y_A^*(z) = \frac{T^2 z^{-1} + T^2 z^{-2}}{(2+PT) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC3C
C
C   Q(T)=RAMP  &  P(T)=T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = T**2
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('   T=',7X,'  Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('   XX=',7X,'    PR=')
      END
```

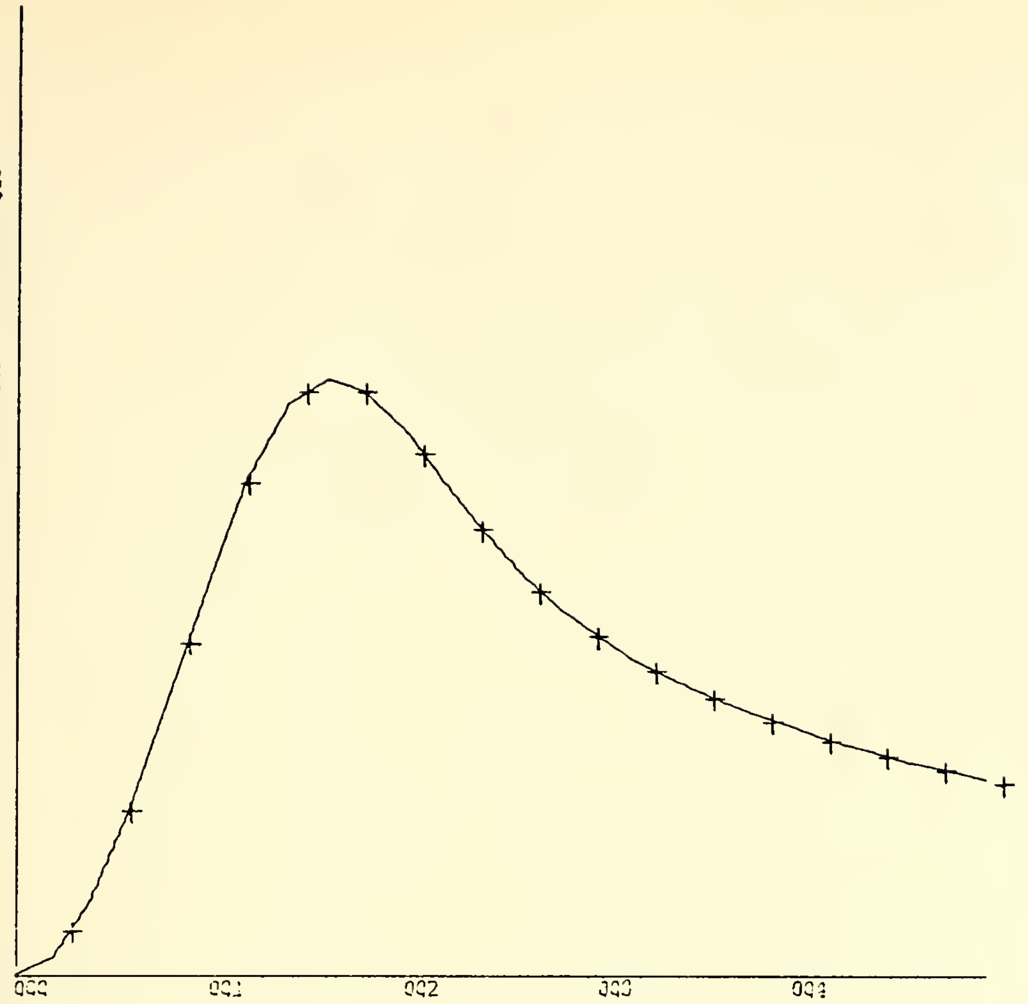
PROGRAM 6




x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

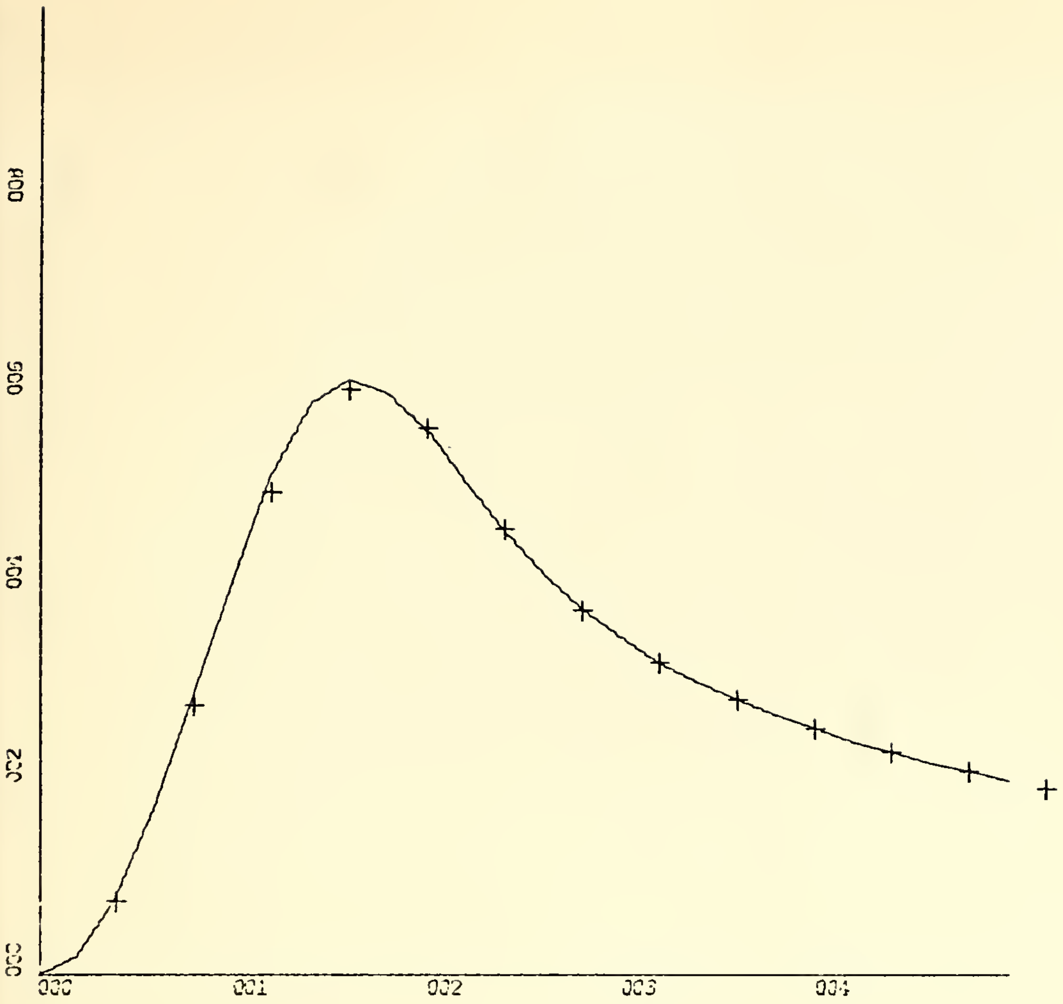
FIGURE 76



FOR DELTA T = 0.3 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.3 | 0.04440 |
| 0.6 | 0.16964 |
| 0.9 | 0.34372 |
| 1.2 | 0.50736 |
| 1.5 | 0.60020 |
| 1.8 | 0.60070 |
| 2.1 | 0.53792 |
| 2.4 | 0.45981 |
| 2.7 | 0.39529 |
| 3.0 | 0.34810 |
| 3.3 | 0.31257 |
| 3.6 | 0.28430 |
| 3.9 | 0.26104 |
| 4.2 | 0.24149 |
| 4.5 | 0.22477 |
| 4.8 | 0.21028 |
| 5.1 | 0.19760 |

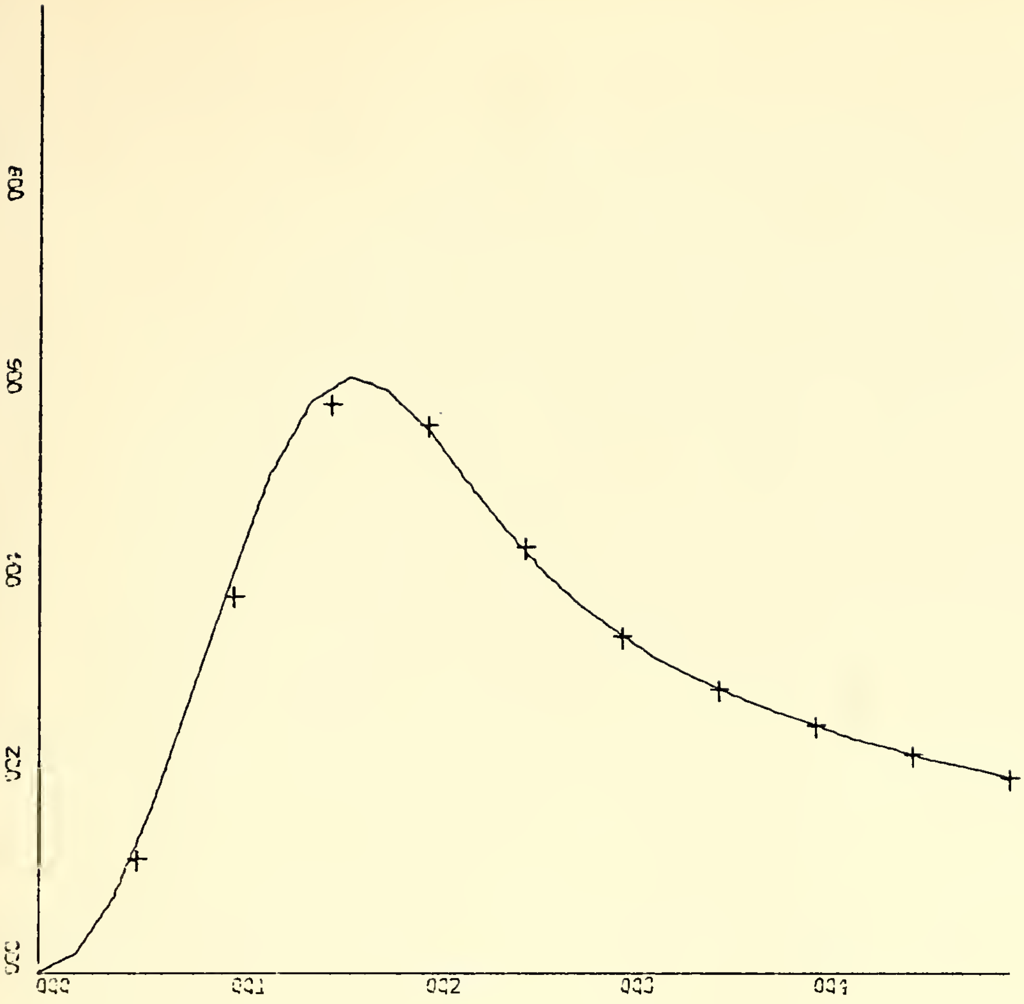
FIGURE 77



FOR DELTA T = 0.4 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.4 | 0.07752 |
| 0.8 | 0.27929 |
| 1.2 | 0.49964 |
| 1.6 | 0.60565 |
| 2.0 | 0.56420 |
| 2.4 | 0.46136 |
| 2.8 | 0.37768 |
| 3.2 | 0.32332 |
| 3.6 | 0.28429 |
| 4.0 | 0.25415 |
| 4.4 | 0.23007 |
| 4.8 | 0.21028 |
| 5.2 | 0.19371 |

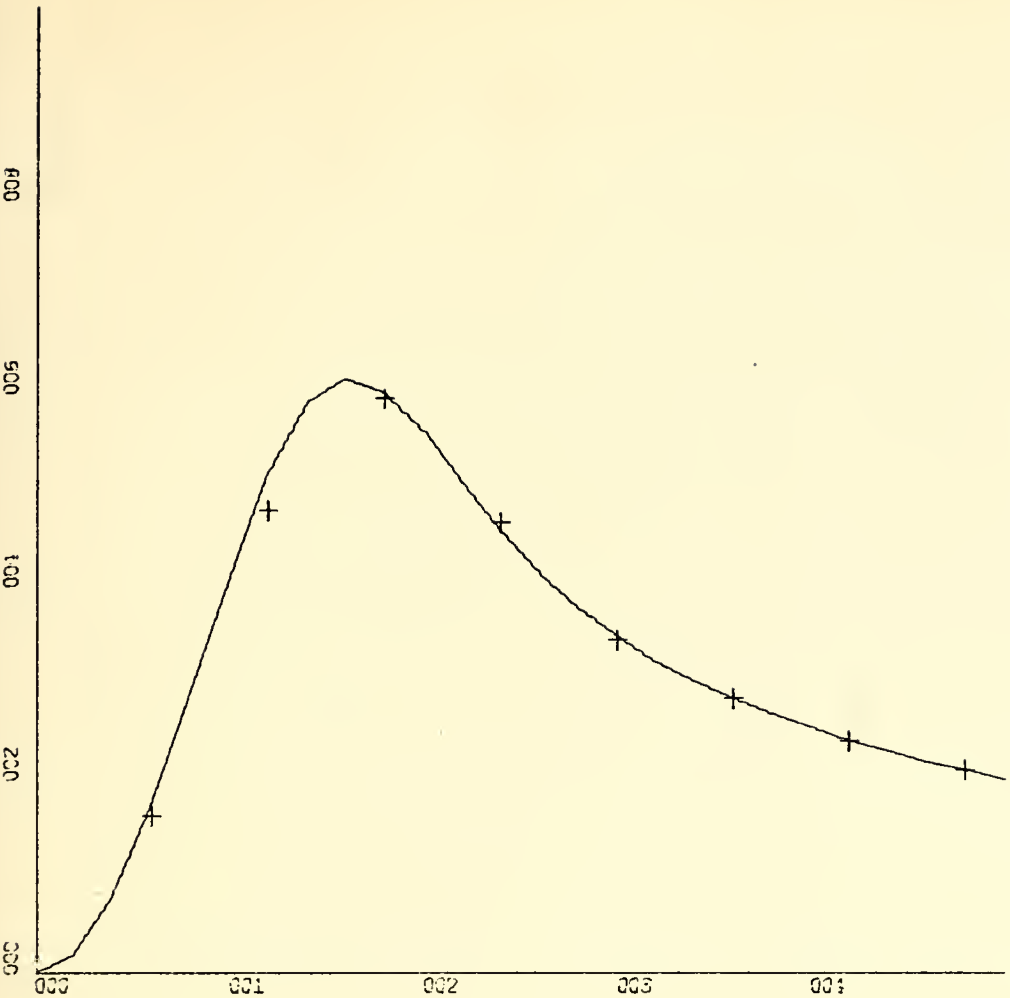
FIGURE 78



FOR DELTA T = 0.5    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.5 | 0.11765 |
| 1.0 | 0.38824 |
| 1.5 | 0.58635 |
| 2.0 | 0.56576 |
| 3.0 | 0.34709 |
| 3.5 | 0.29320 |
| 4.0 | 0.25405 |
| 4.5 | 0.22480 |
| 5.0 | 0.20162 |

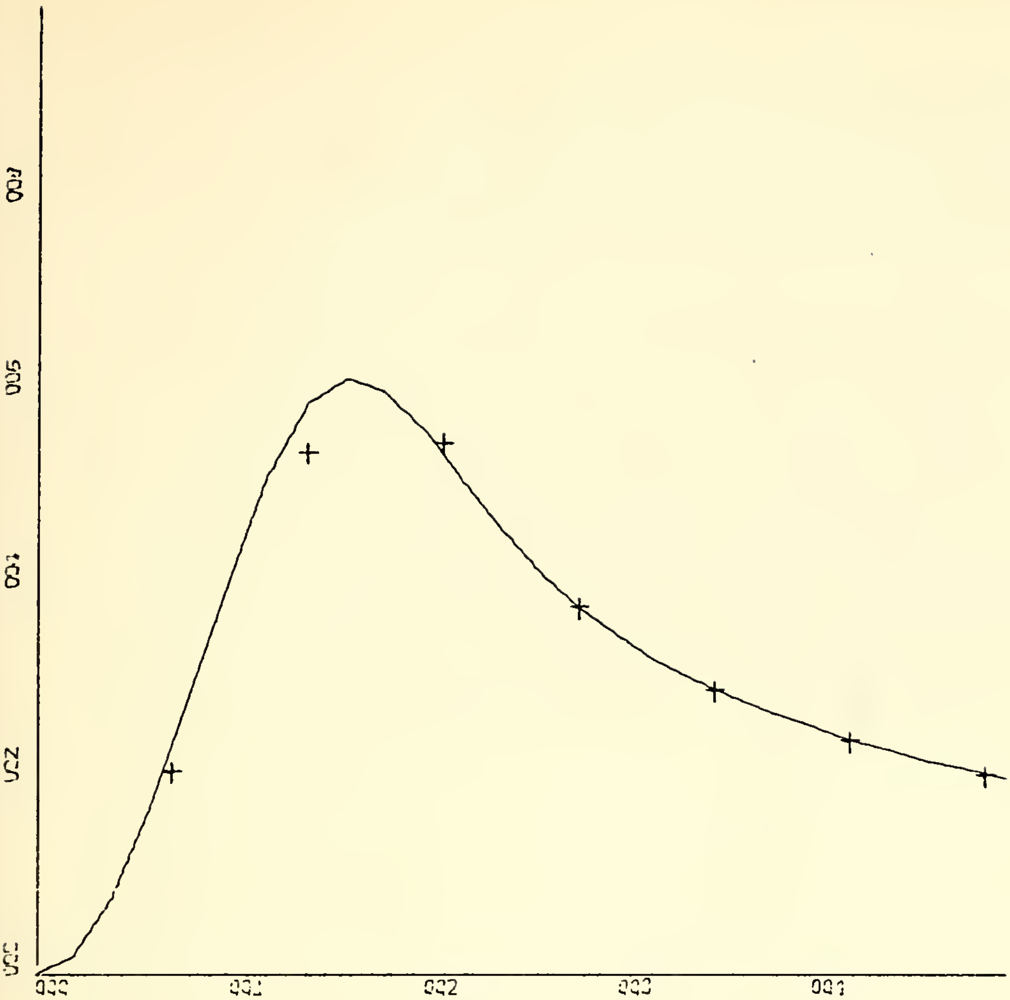
FIGURE 79



FOR DELTA T = 0.6    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.6 | 0.16245 |
| 1.2 | 0.47829 |
| 1.8 | 0.59415 |
| 2.4 | 0.46797 |
| 3.0 | 0.34576 |
| 3.6 | 0.28482 |
| 4.2 | 0.24117 |
| 4.8 | 0.21043 |

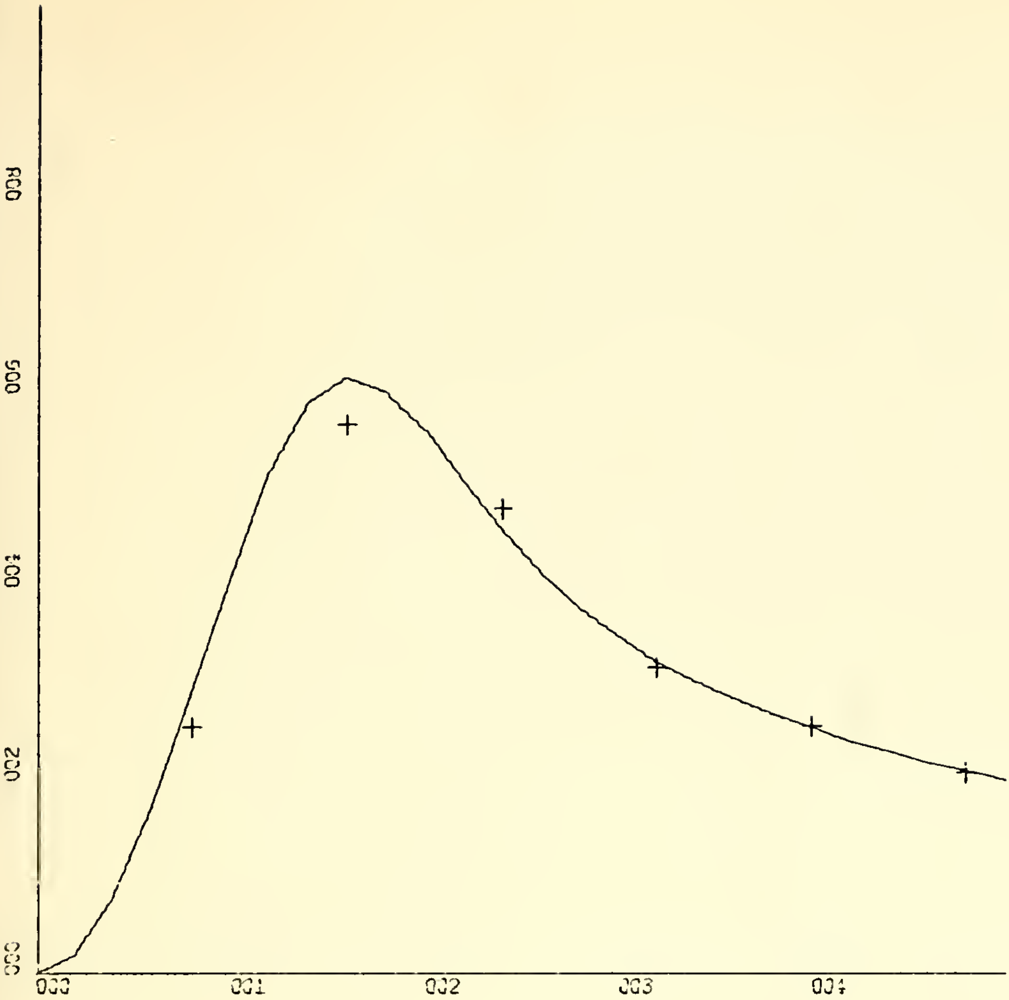
FIGURE 80



FOR DELTA T = 0.7    TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.7 | 0.20913 |
| 1.4 | 0.53871 |
| 2.1 | 0.54812 |
| 2.8 | 0.37850 |
| 3.5 | 0.29218 |
| 4.2 | 0.24177 |
| 4.9 | 0.20568 |

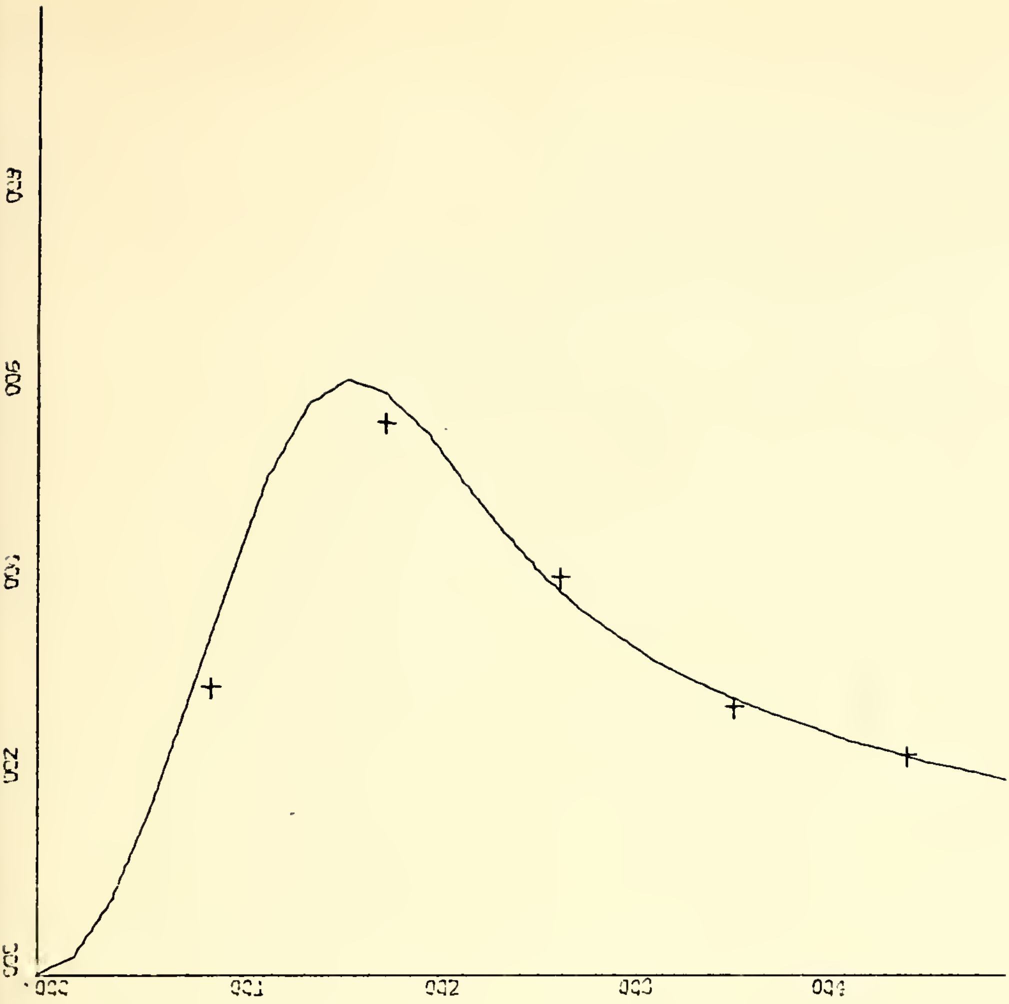
FIGURE 81



FOR DELTA T = 0.8   TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.8 | 0.25478 |
| 1.6 | 0.56796 |
| 2.4 | 0.48014 |
| 3.2 | 0.31670 |
| 4.0 | 0.25669 |
| 4.8 | 0.20888 |

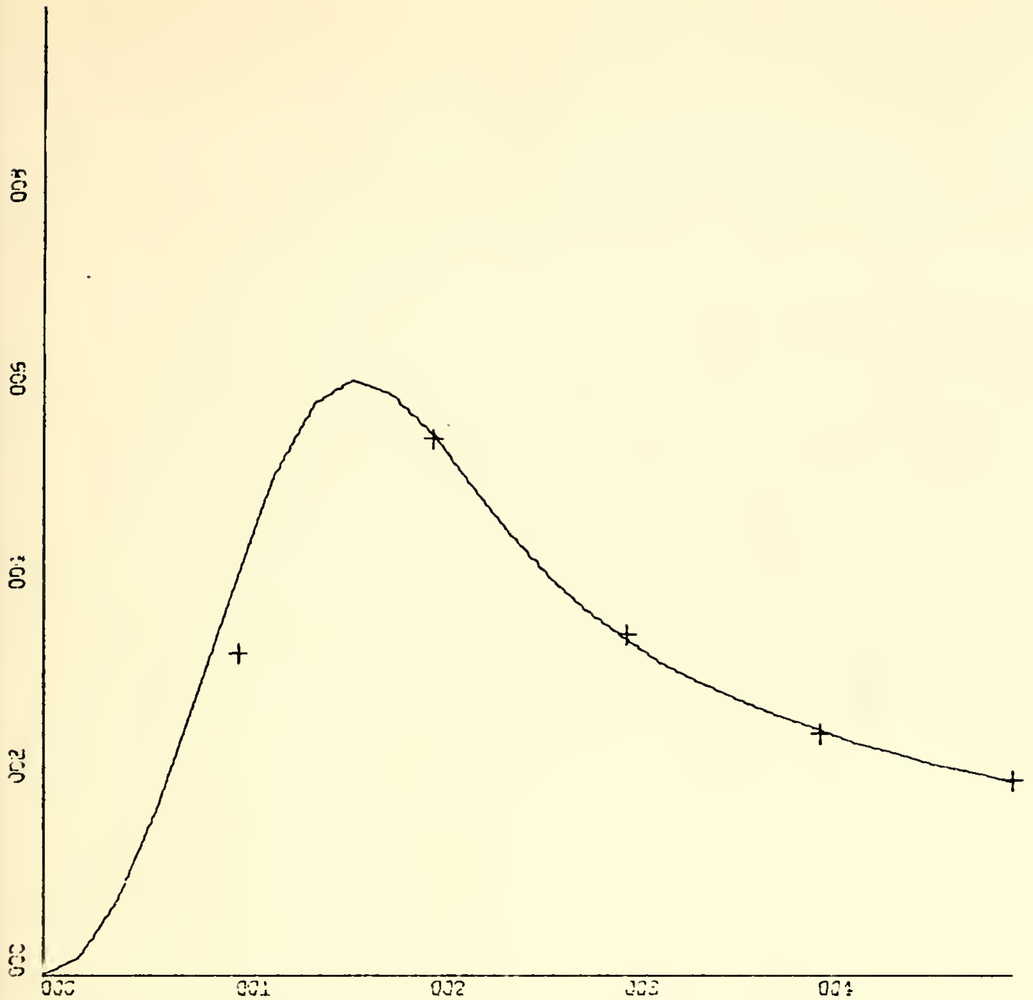
FIGURE 82



FOR DELTA T = 0.9 TFIN = 5.2

| T | Q(N) |
|---|---|
| 0.9 | 0.29681 |
| 1.8 | 0.57104 |
| 2.7 | 0.41198 |
| 3.6 | 0.27744 |
| 4.5 | 0.22788 |

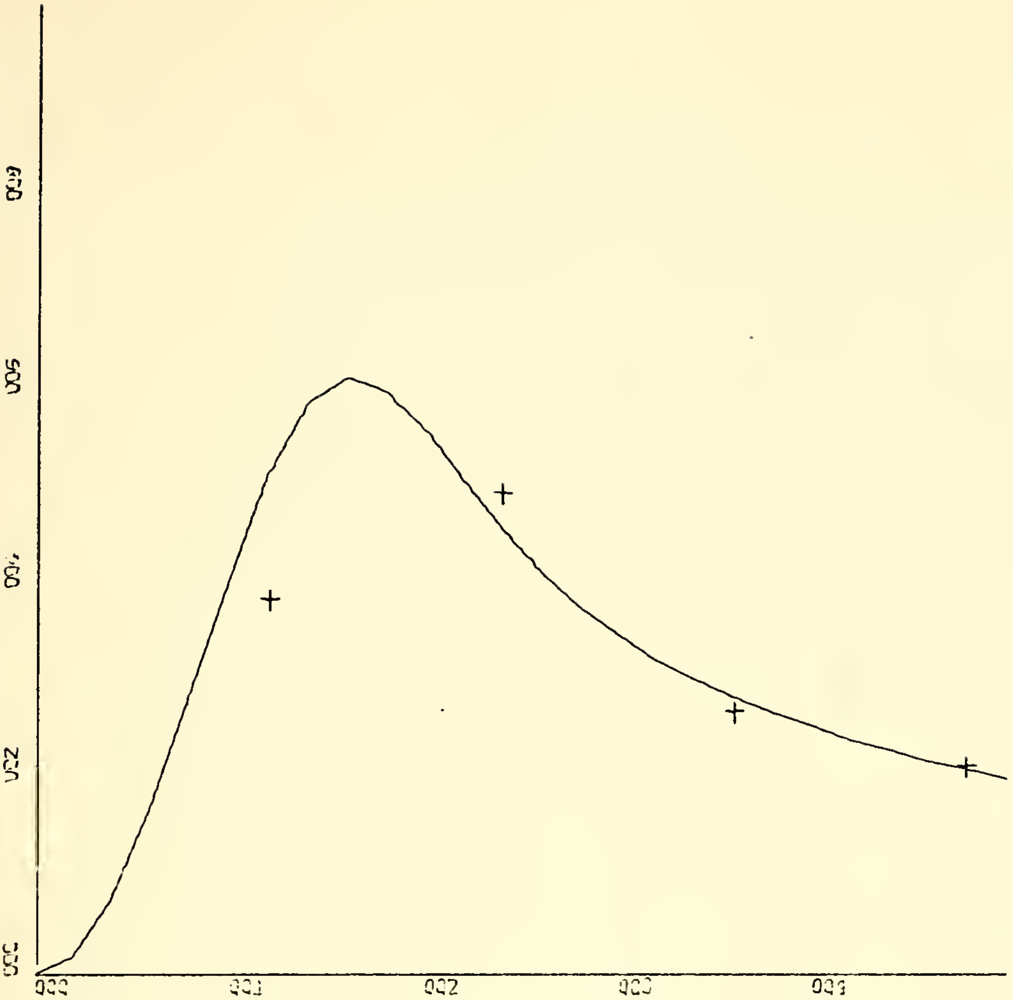
FIGURE 83



FOR DELTA T = 1.0    TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.0 | 0.33333 |
| 2.0 | 0.55556 |
| 3.0 | 0.35354 |
| 4.0 | 0.25140 |
| 5.0 | 0.20298 |

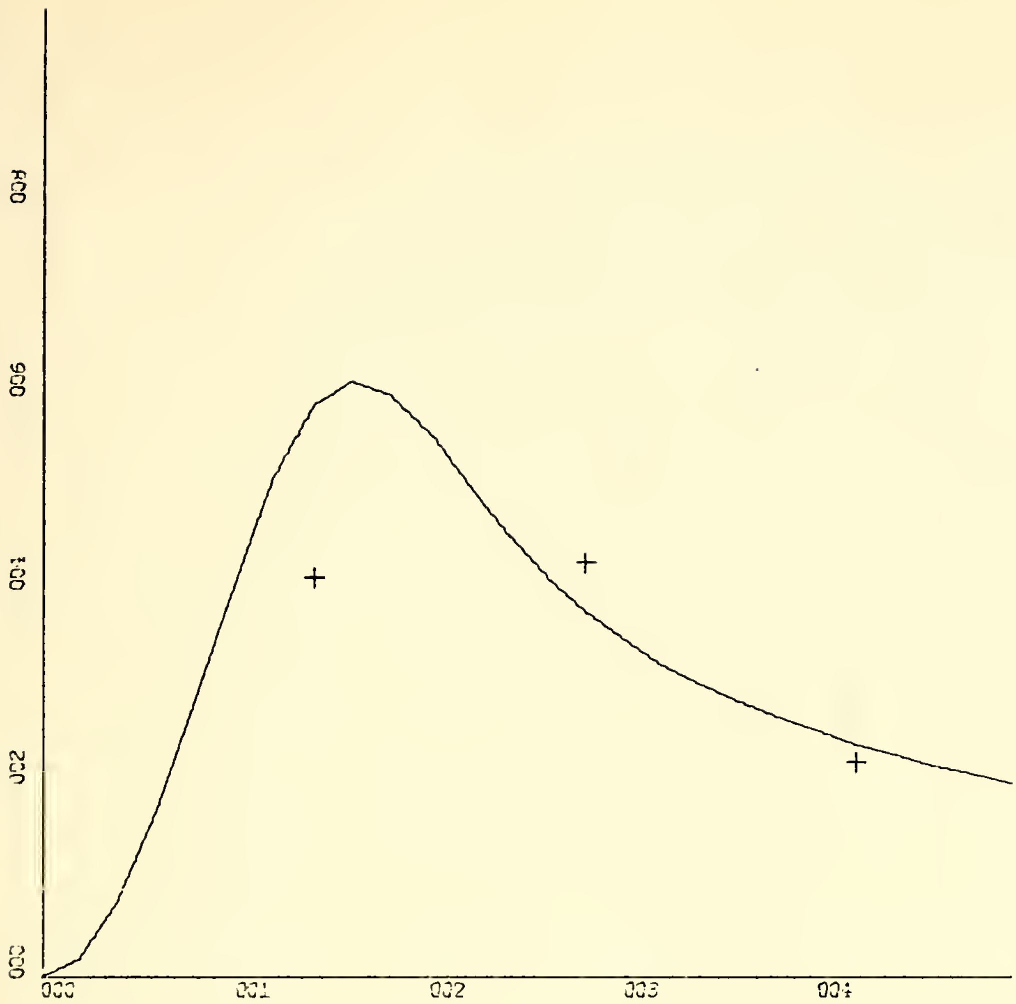
FIGURE 84



FOR DELTA T = 1.2 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.2 | 0.38627 |
| 2.4 | 0.49653 |
| 3.6 | 0.27125 |
| 4.8 | 0.21600 |

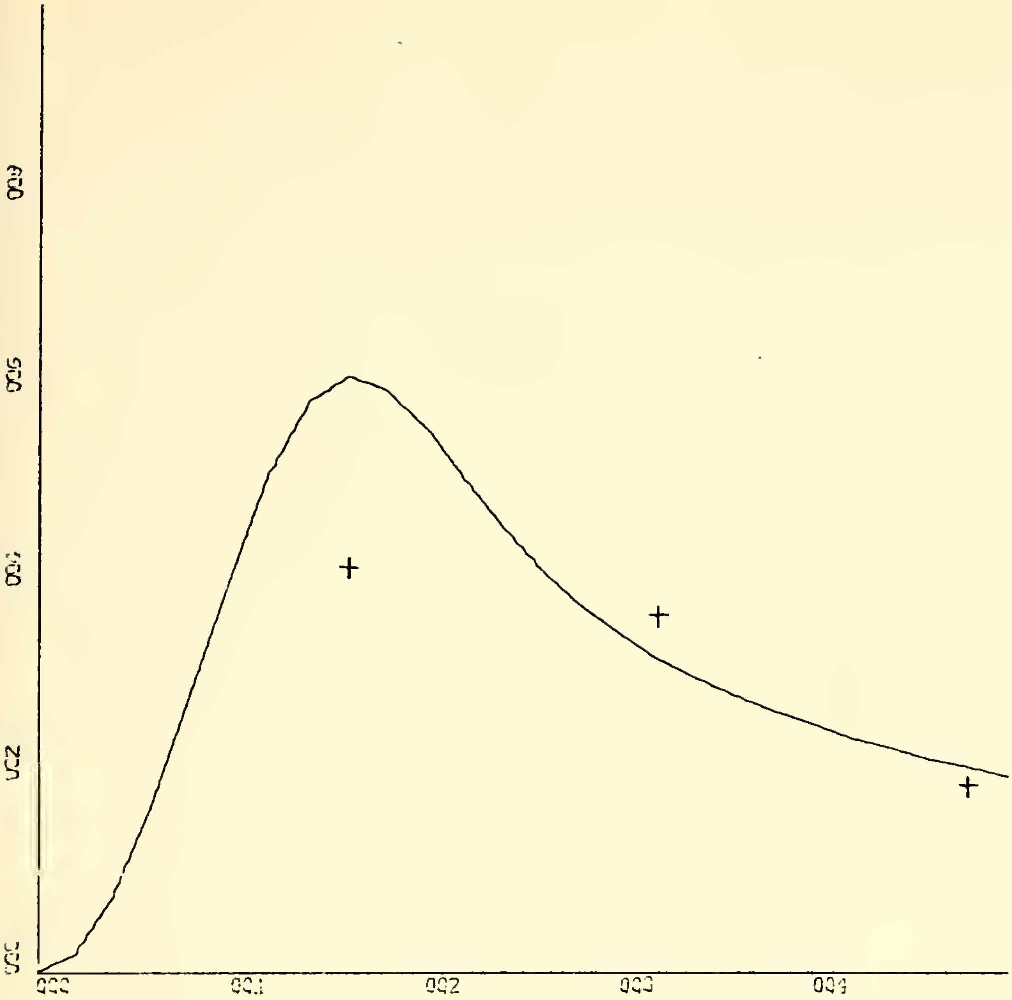
FIGURE 85



FOR DELTA T = 1.4   TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.4 | 0.41315 |
| 2.8 | 0.42946 |
| 4.2 | 0.22270 |

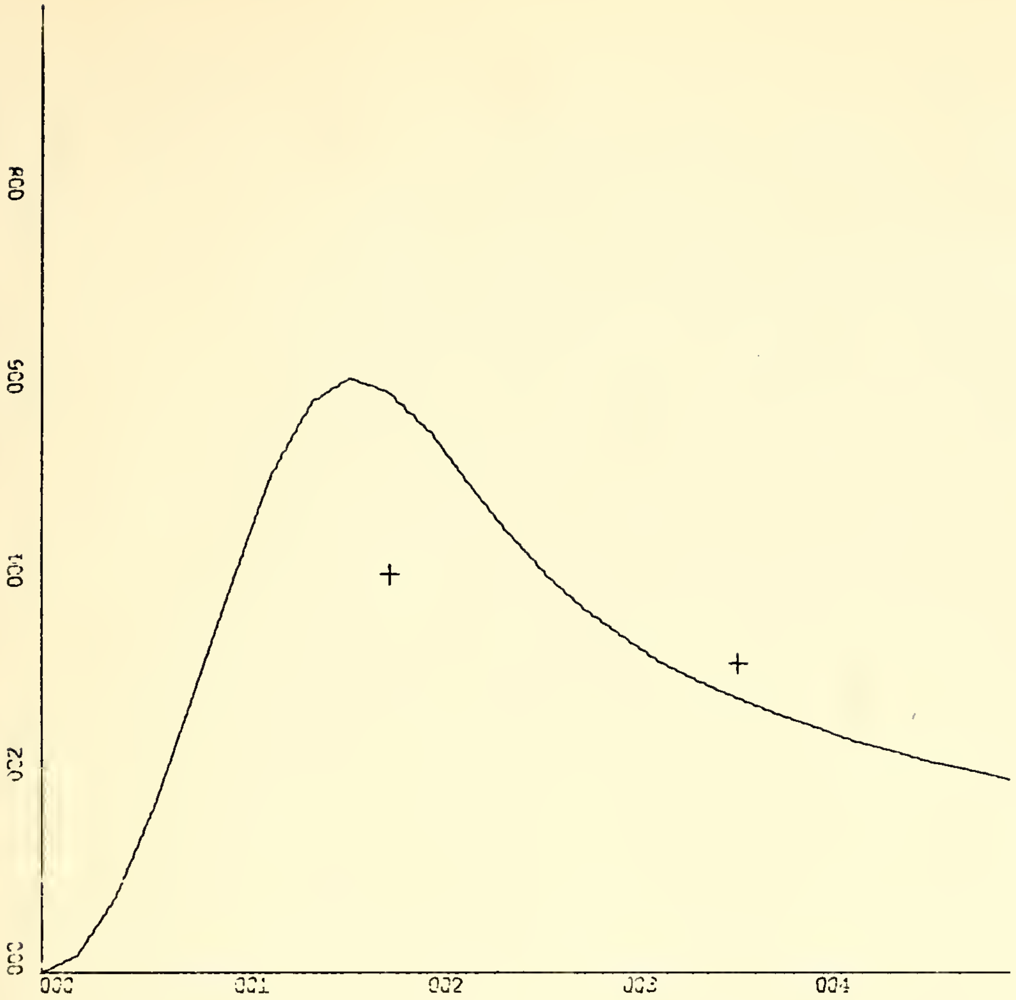
FIGURE 86



FOR DELTA T = 1.6 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.6 | 0.41995 |
| 3.2 | 0.36988 |
| 4.8 | 0.19246 |

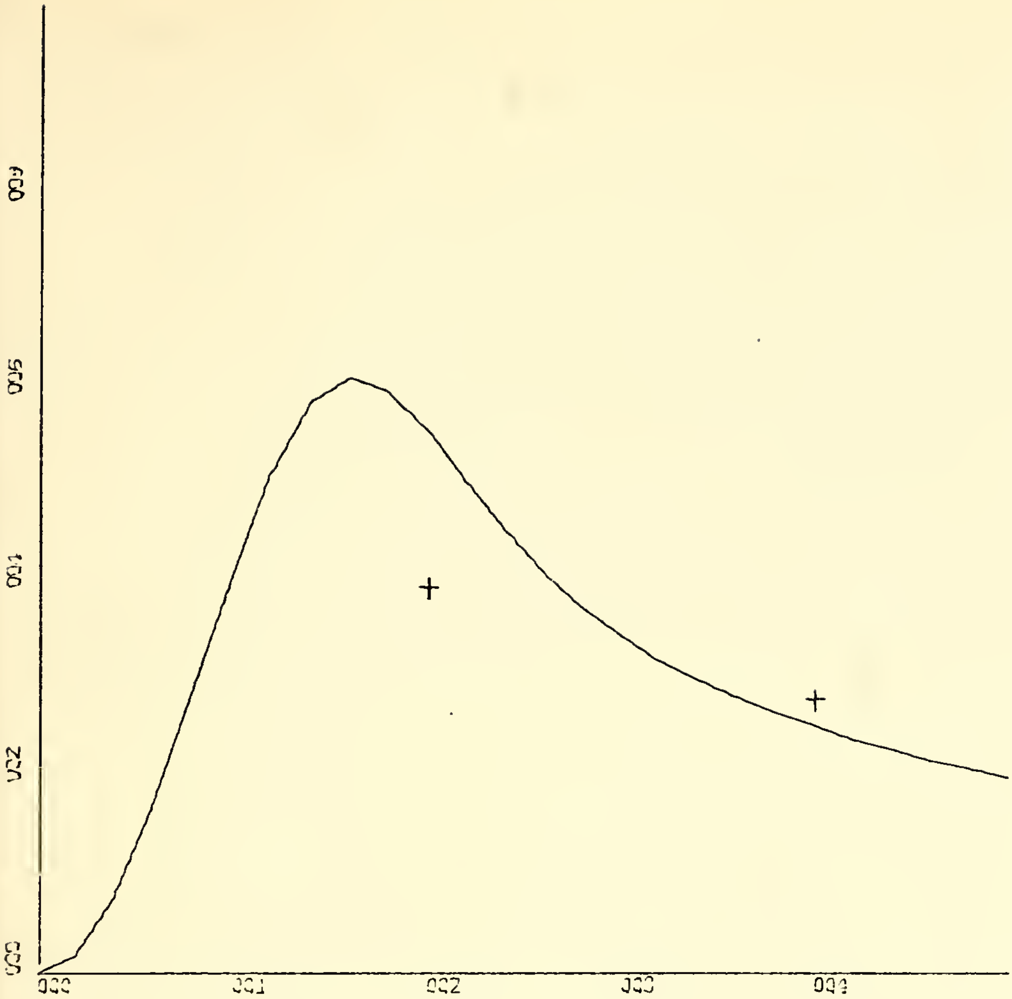
FIGURE 87



FOR DELTA T = 1.8 TFIN = 5.2

| T | Q(N) |
|---|---|
| 1.8 | 0.41369 |
| 3.6 | 0.32118 |

FIGURE 88



FOR DELTA T = 2.0 TFIN = 5.2

| T | Q(N) |
|---|---|
| 2.0 | 0.40000 |
| 4.0 | 0.28235 |

FIGURE 89

D. CASE 7 — $\dot{Y} + \sqrt{t}\,Y = t^2$

Following the procedure as in II C:

$$\frac{dY}{dt} + PY = t^2 \qquad\qquad P = \sqrt{t}$$

The Laplace transform in powers of $s^{-k}$:

$$sy + Py = 2/s^3 \qquad\qquad y = \frac{2s^{-4}}{1 + Ps^{-1}}$$

Substituting the z-forms of Table I into the expression above yields:

$$y_A^*(z) = \frac{\dfrac{T^4}{6} \cdot \dfrac{z^{-1} + 4z^{-2} + z^{-3}}{(1 - z^{-1})^4} - \dfrac{T^4}{720}}{1 + P\,\dfrac{T}{2} \cdot \dfrac{(1 + z)^{-1}}{(1 - z)^{-1}}}$$

Dividing by T and rearranging in crescent powers of $z^{-1}$:

$$y_A^*(z) = \frac{-T^3 + (124T^3)z^{-1} + (474T^3)z^{-2} + 124T^3z^{-3} = T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+ \cdots + (360-180PT)z^{-4}} \qquad (47)$$

Choosing for first value for T = 0.2, and substituting in the previous equation, the denominator is divided into the numerator, after the value of P is replaced in the expression above for the corresponding time.

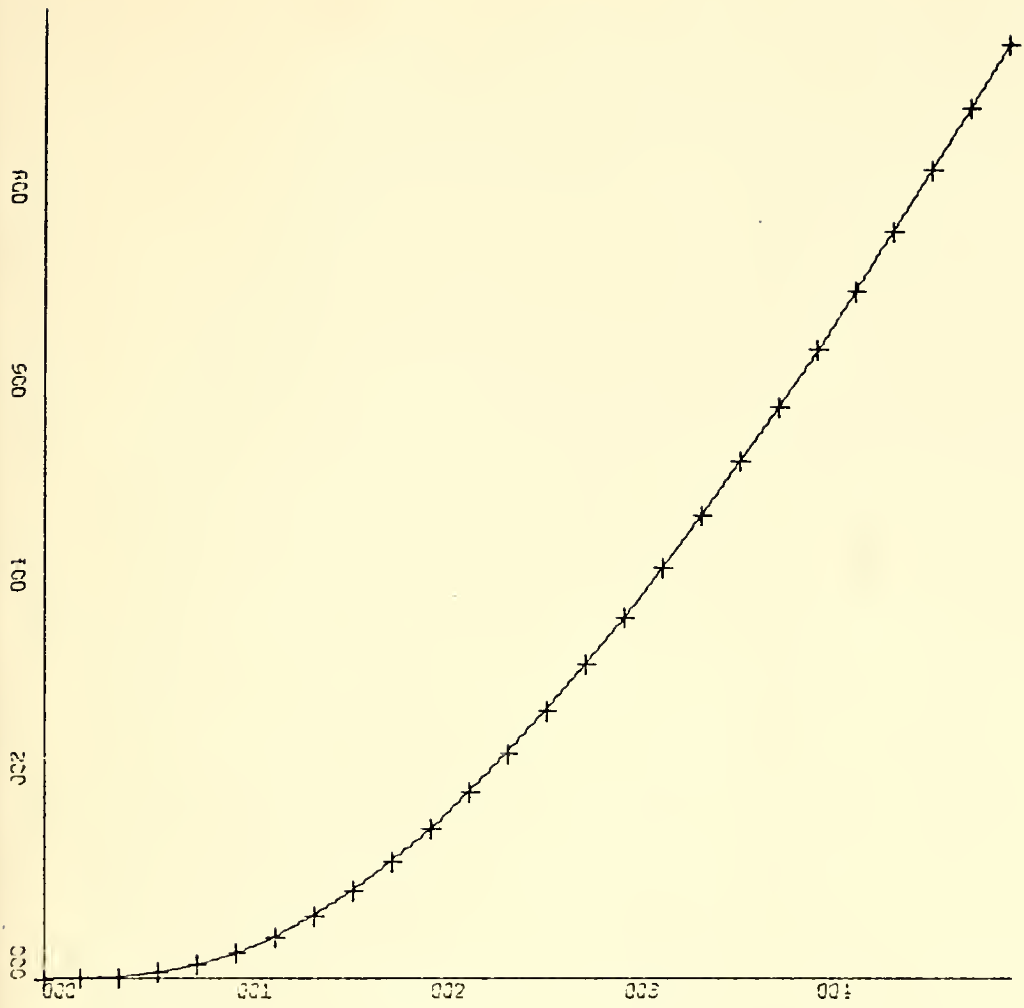
The results and the plots for the precise and approximated solution follow, with iteration times from 0.2 sec. to 2.0 sec. (Figs. 90 to 103). Computer algorithm for solution of this problem is in Program 7 (page 128).

DELTA T = 0.2 TFIN = 5.2

| T | Q(N) | PR |
|---|---|---|
| 0.0 | -0.00002 | 0.0 |
| 0.2 | 0.00255 | 0.00261 |
| 0.4 | 0.01985 | 0.02018 |
| 0.6 | 0.06428 | 0.06511 |
| 0.8 | 0.14500 | 0.14648 |
| 1.0 | 0.26791 | 0.27012 |
| 1.2 | 0.43602 | 0.43897 |
| 1.4 | 0.65001 | 0.65363 |
| 1.6 | 0.90881 | 0.91298 |
| 1.8 | 1.21016 | 1.21470 |
| 2.0 | 1.55110 | 1.55590 |
| 2.2 | 1.92838 | 1.93330 |
| 2.4 | 2.33869 | 2.34370 |
| 2.6 | 2.77893 | 2.78380 |
| 2.8 | 3.24627 | 3.25100 |
| 3.0 | 3.73821 | 3.74280 |
| 3.2 | 4.25262 | 4.25700 |
| 3.4 | 4.78767 | 4.79180 |
| 3.6 | 5.34186 | 5.34590 |
| 3.8 | 5.91392 | 5.91780 |
| 4.0 | 6.50283 | 6.50670 |
| 4.2 | 7.10775 | 7.11160 |
| 4.4 | 7.72795 | 7.73200 |
| 4.6 | 8.36285 | 8.36720 |
| 4.8 | 9.01195 | 9.01680 |
| 5.0 | 9.67481 | 9.68030 |

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T) = T**2  &  P(T) =SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = SQRT(T)
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT (' T=',8X,' Q(N)=')
    8 FORMAT ('   XX=',7X,'   PR=')
      END
```

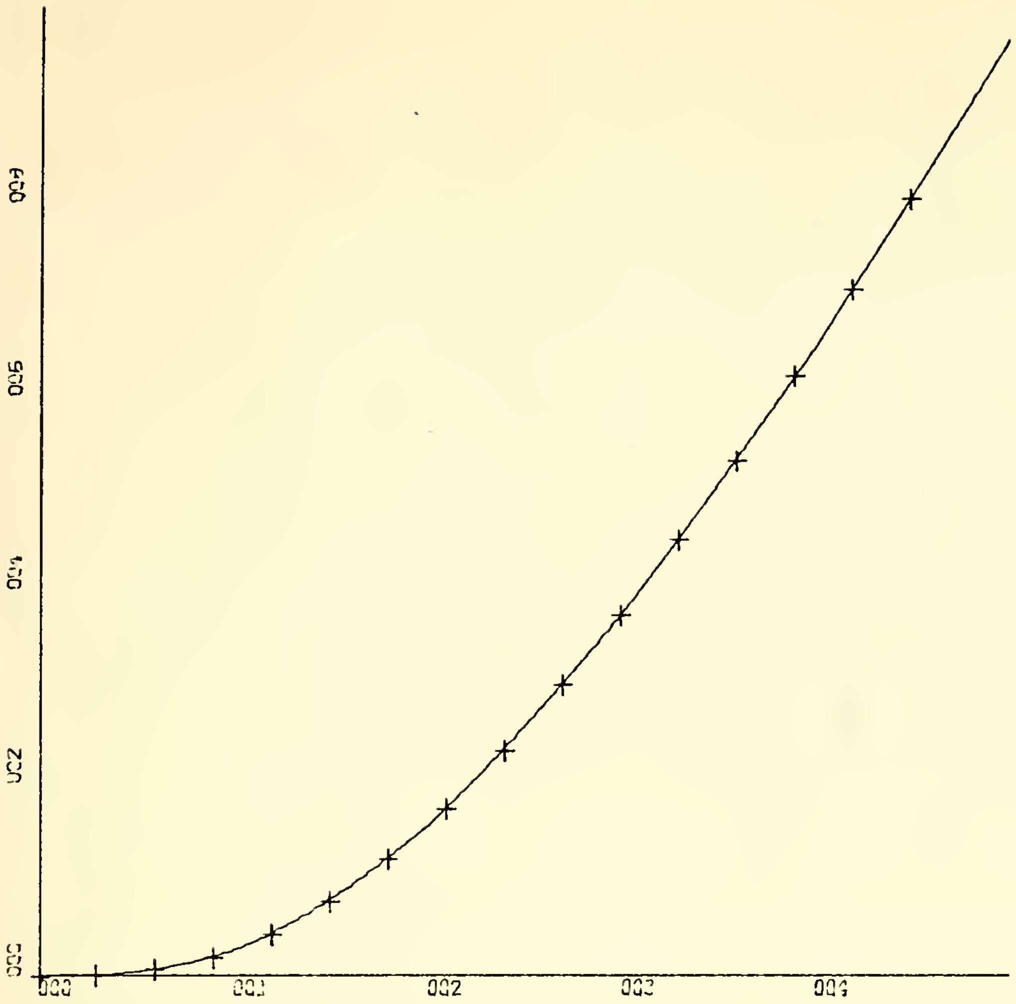
PROGRAM 7



x-scale = 1.0 units/inch
y-scale = 2.0 units/inch

FOR DELTA T = 0.2

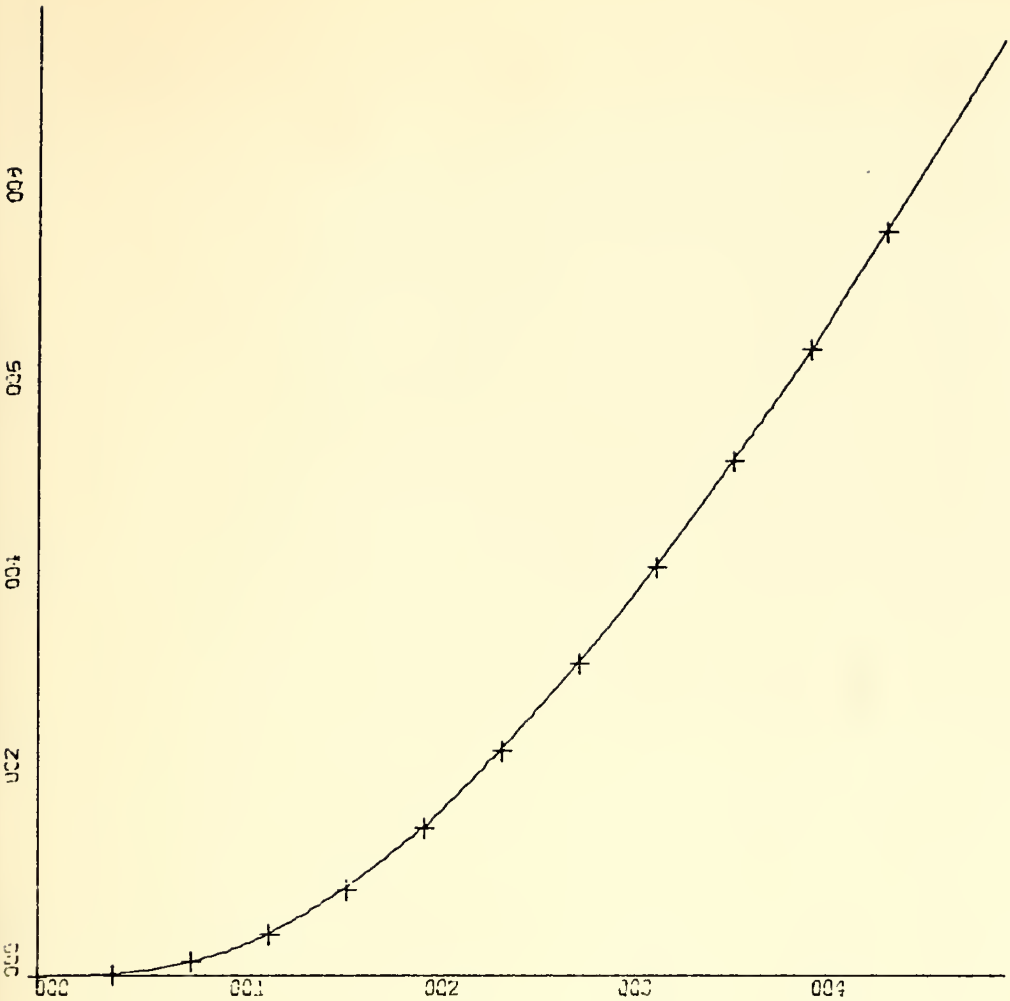
FIGURE 90



FOR DELTA T = 0.3 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00007 |
| 0.3 | 0.00832 |
| 0.6 | 0.06328 |
| 0.9 | 0.19866 |
| 1.2 | 0.43235 |
| 1.5 | 0.76902 |
| 1.8 | 1.20437 |
| 2.1 | 1.72913 |
| 2.4 | 2.33229 |
| 2.7 | 3.00312 |
| 3.0 | 3.73225 |
| 3.3 | 4.51208 |
| 3.6 | 5.33673 |
| 3.9 | 6.20176 |
| 4.2 | 7.10386 |
| 4.5 | 8.04059 |

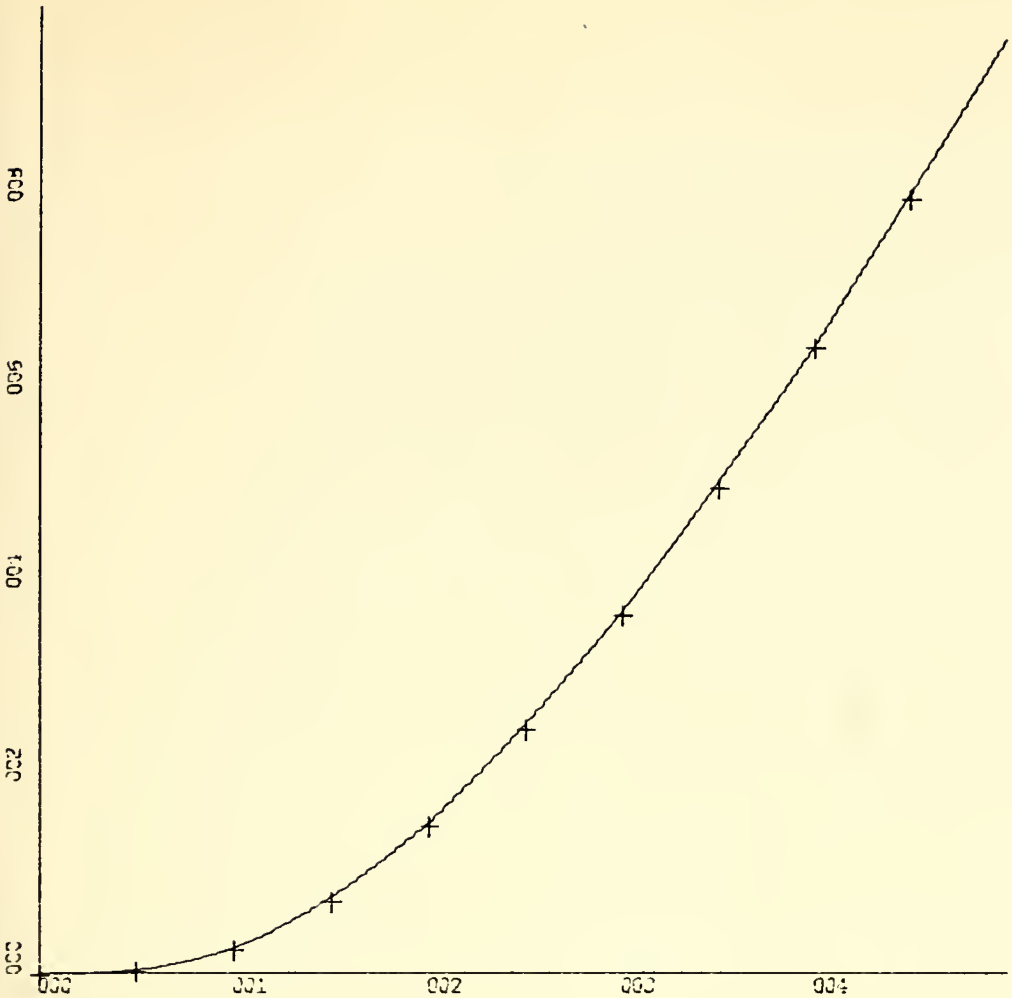
FIGURE 91



FOR DELTA T = 0.4 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00018 |
| 0.4 | 0.01894 |
| 0.8 | 0.14071 |
| 1.2 | 0.42726 |
| 1.6 | 0.89625 |
| 2.0 | 1.53631 |
| 2.4 | 2.32327 |
| 2.8 | 3.23134 |
| 3.2 | 4.23875 |
| 3.6 | 5.32927 |
| 4.0 | 6.49164 |
| 4.4 | 7.71831 |

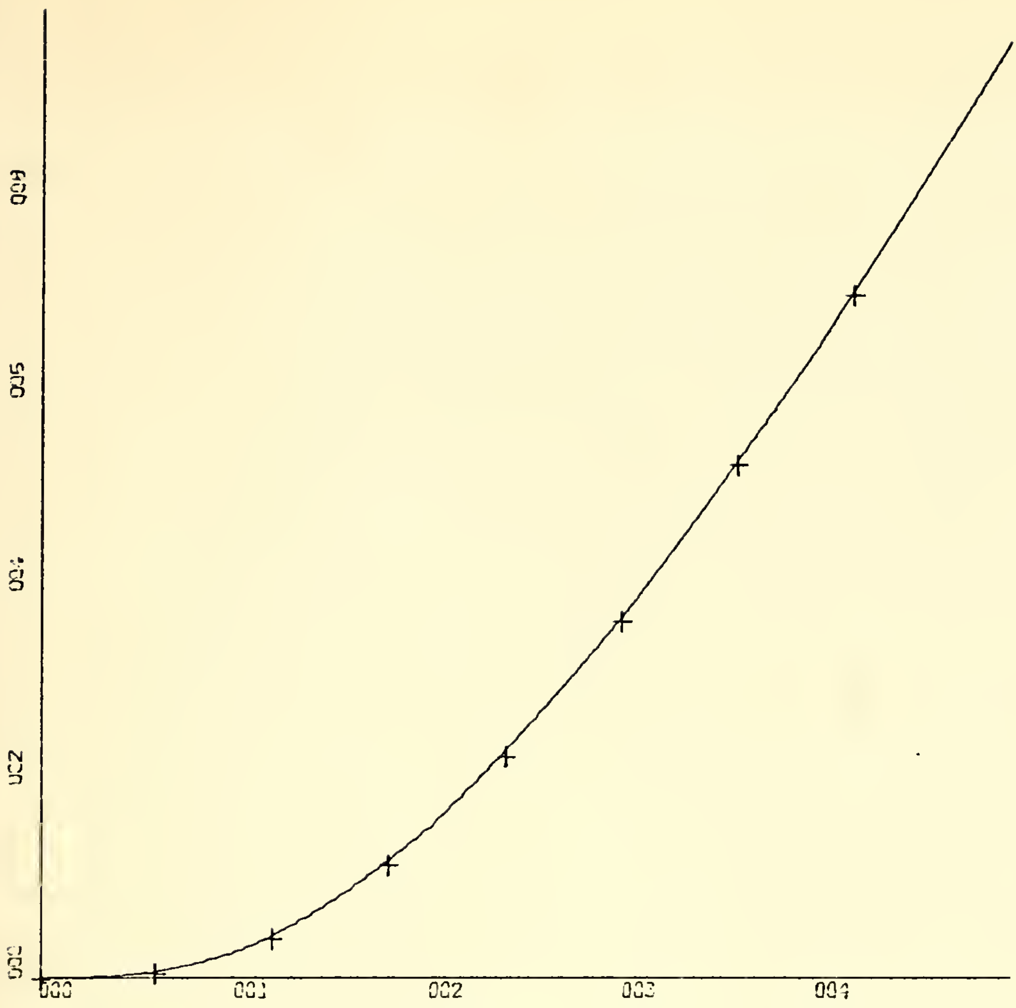
FIGURE 92



FOR DELTA T = 0.5 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00035 |
| 0.5 | 0.03541 |
| 1.0 | 0.25665 |
| 1.5 | 0.75346 |
| 2.0 | 1.52519 |
| 2.5 | 2.52824 |
| 3.0 | 3.71282 |
| 3.5 | 5.03967 |
| 4.0 | 6.48280 |
| 4.5 | 8.02638 |

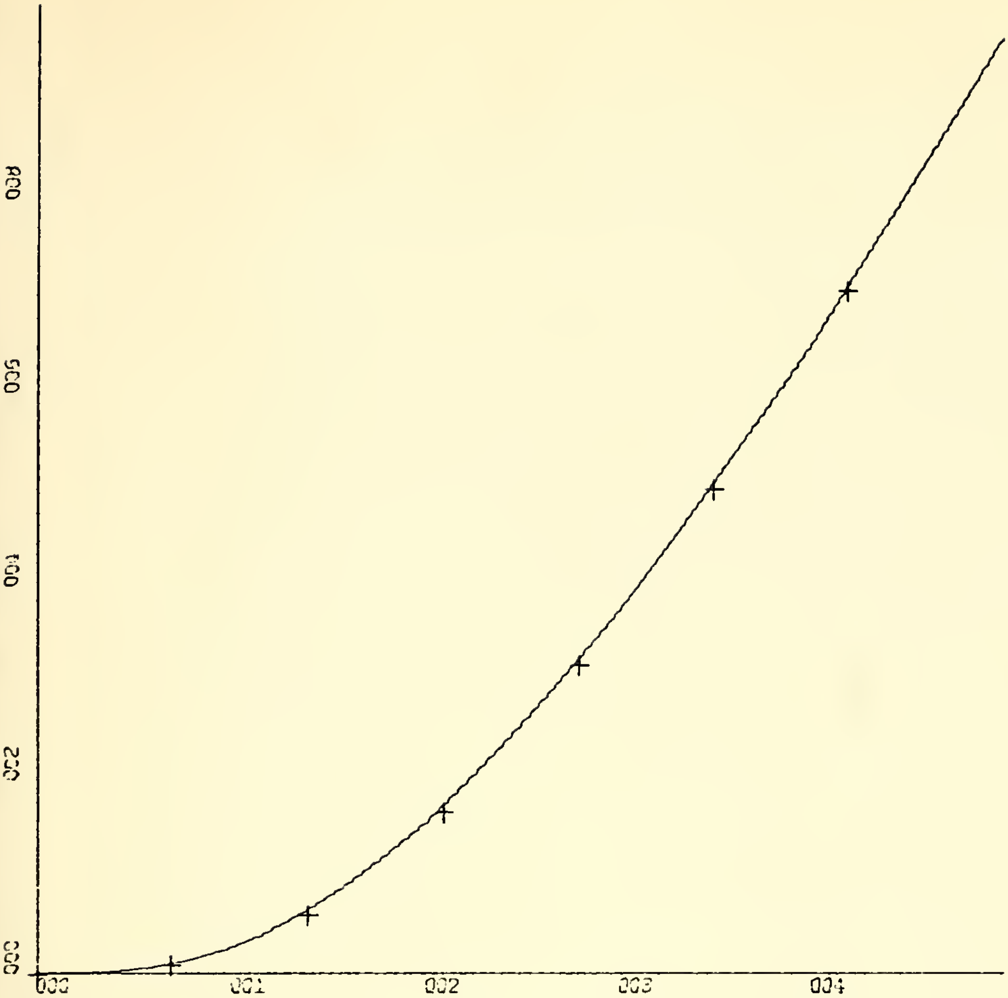
FIGURE 93



FOR DELTA T = 0.6    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00060 |
| 0.6 | 0.05842 |
| 1.2 | 0.41309 |
| 1.8 | 1.17315 |
| 2.4 | 2.29728 |
| 3.0 | 3.69936 |
| 3.6 | 5.30783 |
| 4.2 | 7.07854 |

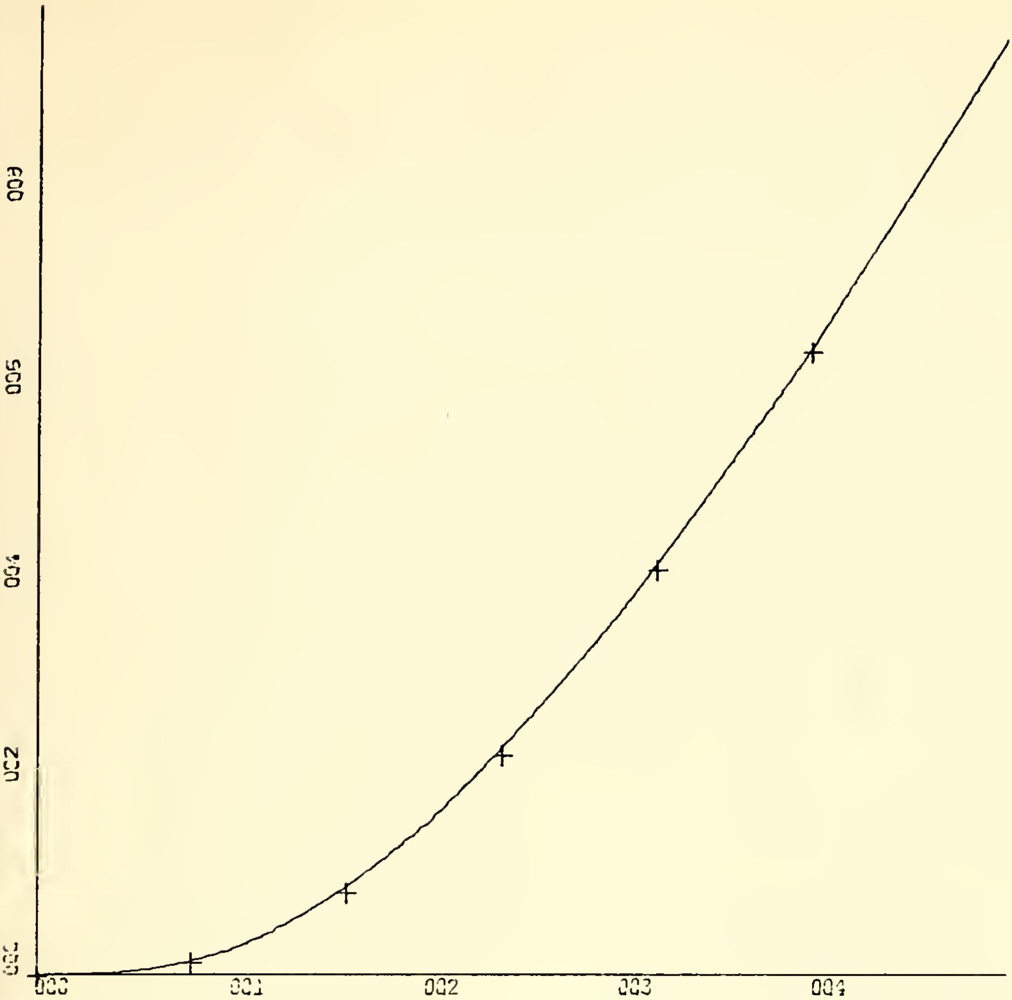
FIGURE 94



FOR DELTA T = 0.7    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00095 |
| 0.7 | 0.08844 |
| 1.4 | 0.61018 |
| 2.1 | 1.67849 |
| 2.8 | 3.18952 |
| 3.5 | 5.01324 |
| 4.2 | 7.06634 |

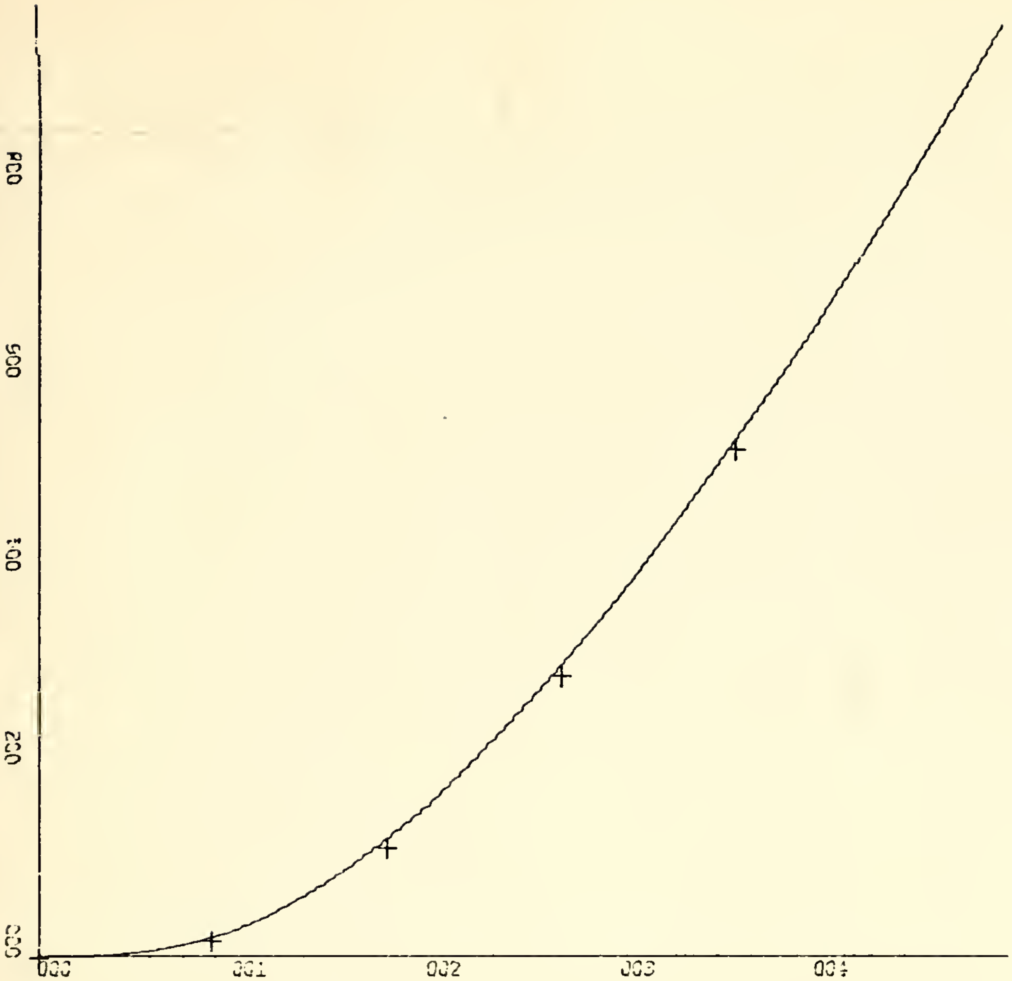
FIGURE 95



FOR DELTA T = 0.8 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00142 |
| 0.8 | 0.12570 |
| 1.6 | 0.84689 |
| 2.4 | 2.26036 |
| 3.2 | 4.18196 |
| 4.0 | 6.44458 |

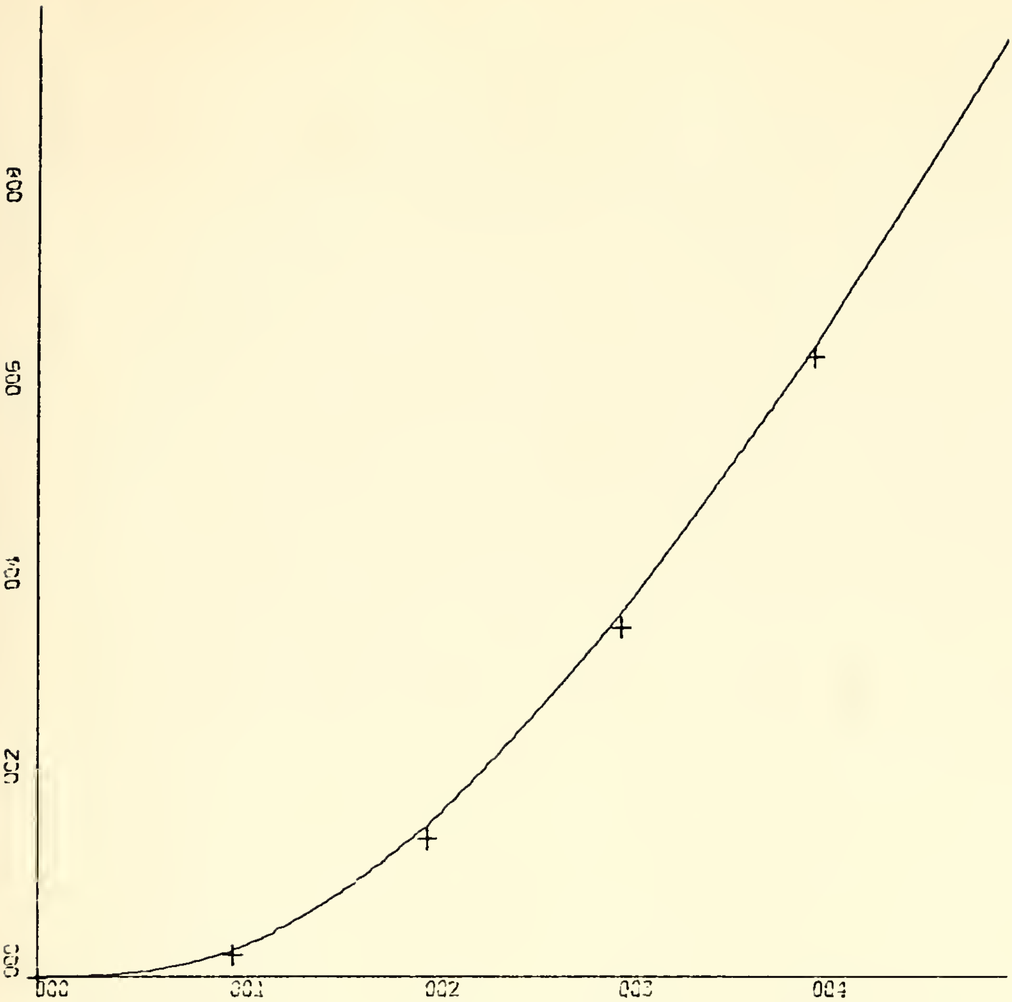
FIGURE 96



FOR DELTA T = 0.9    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00202 |
| 0.9 | 0.17030 |
| 1.8 | 1.12150 |
| 2.7 | 2.90981 |
| 3.6 | 5.25900 |

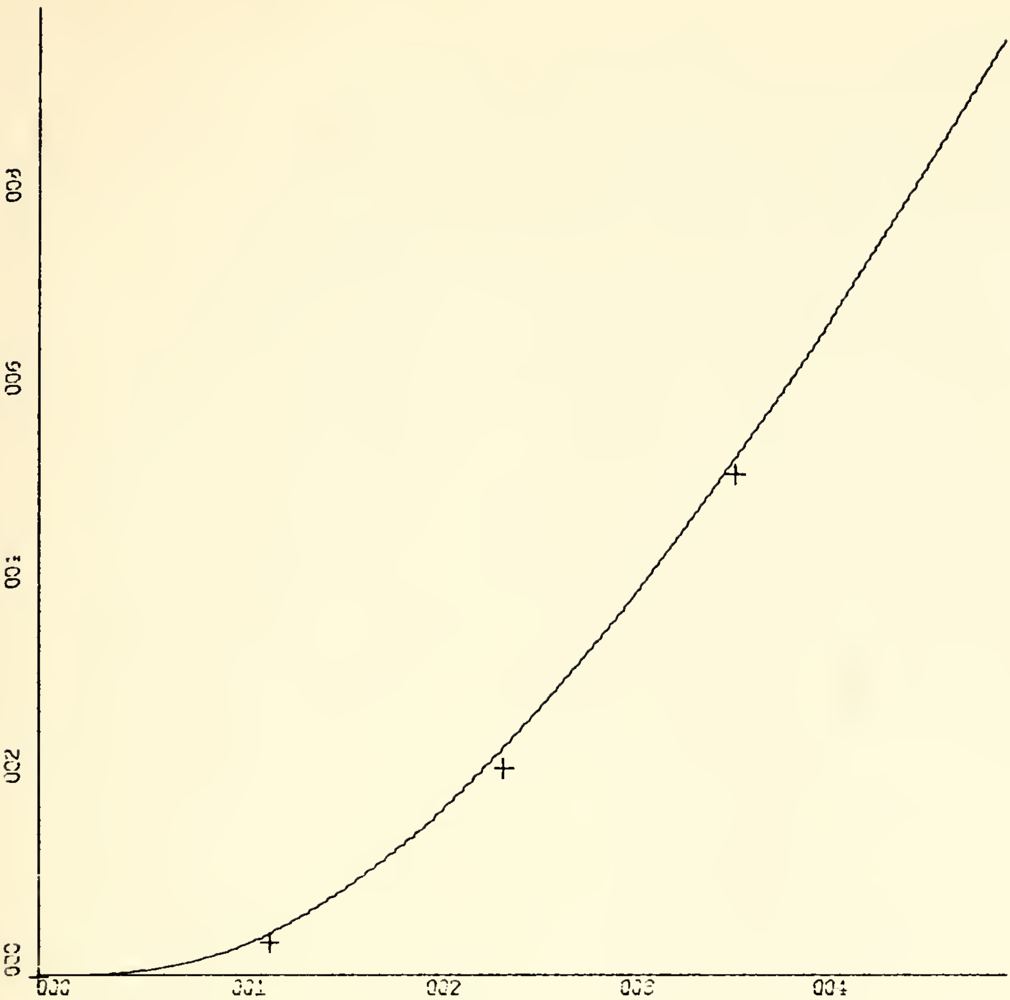
FIGURE 97



FOR DELTA T = 1.0      TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00278 |
| 1.0 | 0.22222 |
| 2.0 | 1.43192 |
| 3.0 | 3.61877 |
| 4.0 | 6.40907 |

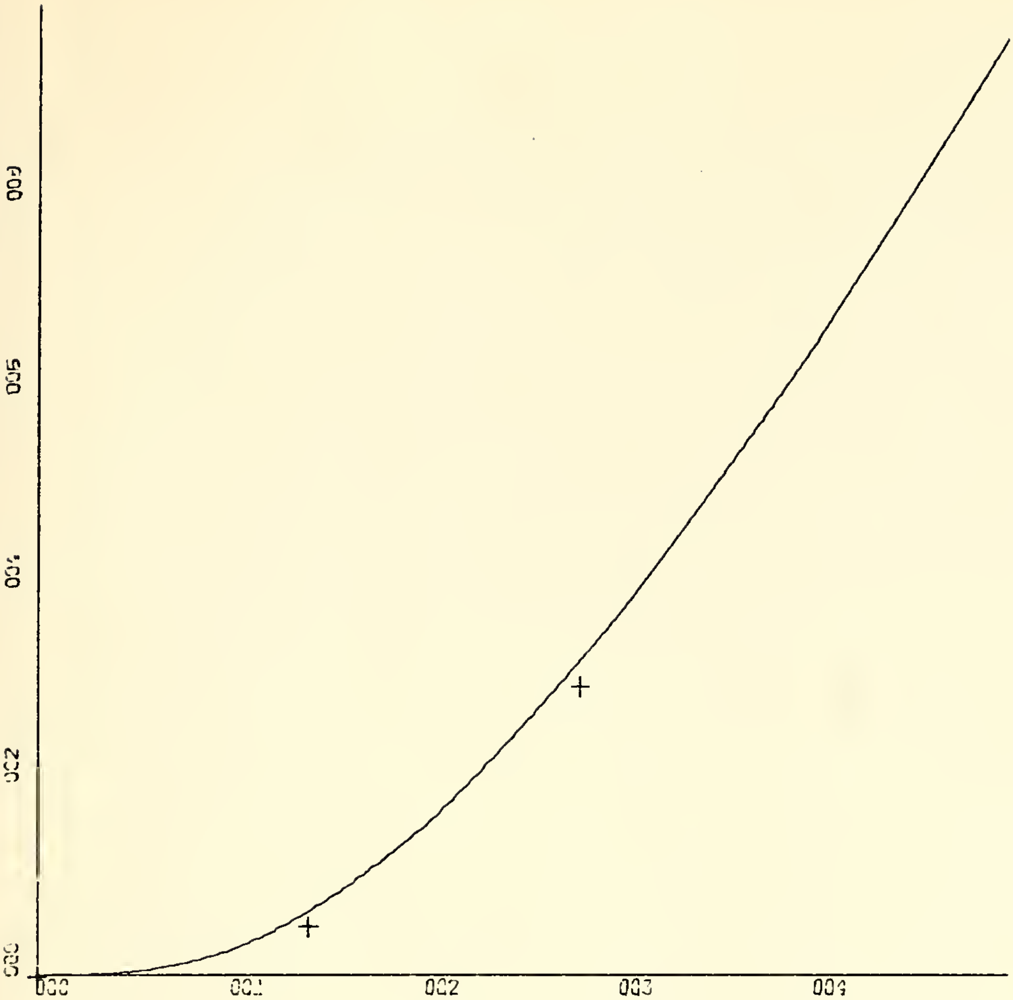
FIGURE 98



FOR DELTA T = 1.2 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00480 |
| 1.2 | 0.34756 |
| 2.4 | 2.15138 |
| 3.6 | 5.18870 |

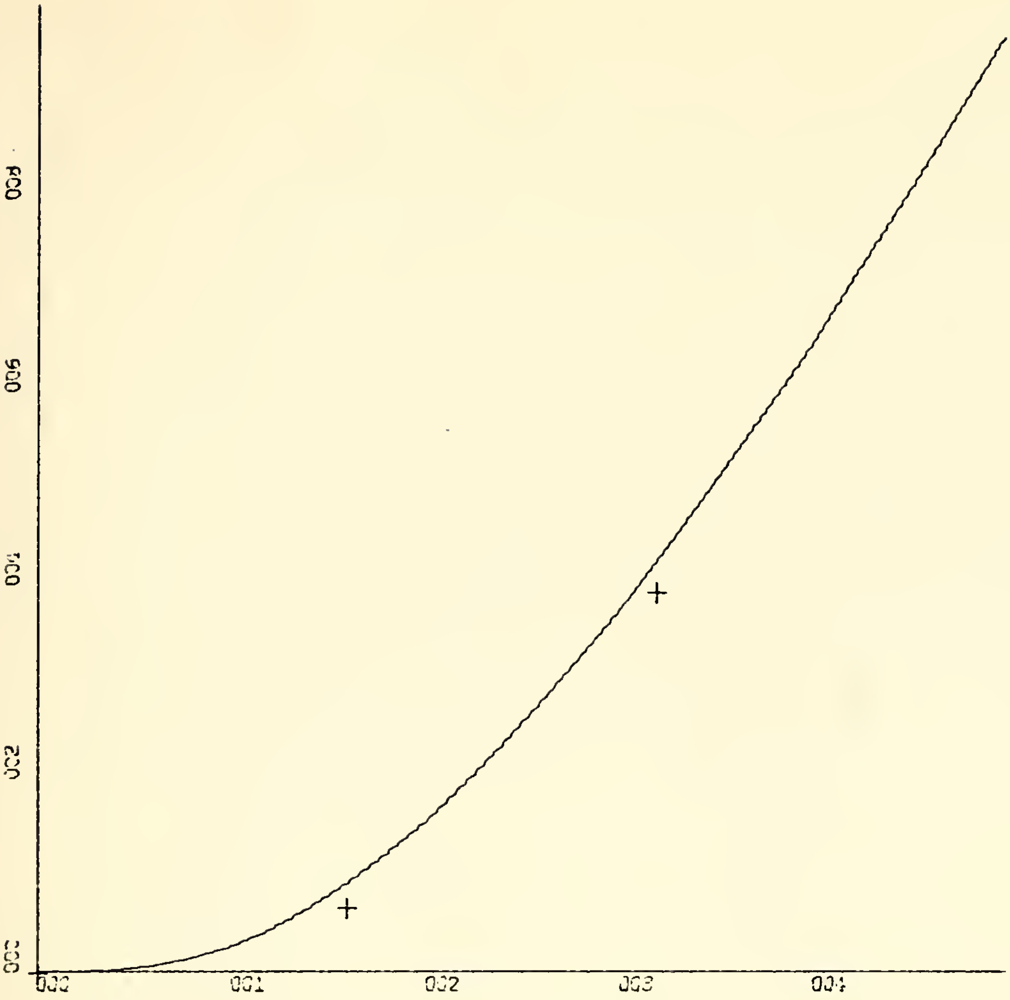
FIGURE 99



FOR DELTA T = 1.4   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00762 |
| 1.4 | 0.50030 |
| 2.8 | 2.98831 |

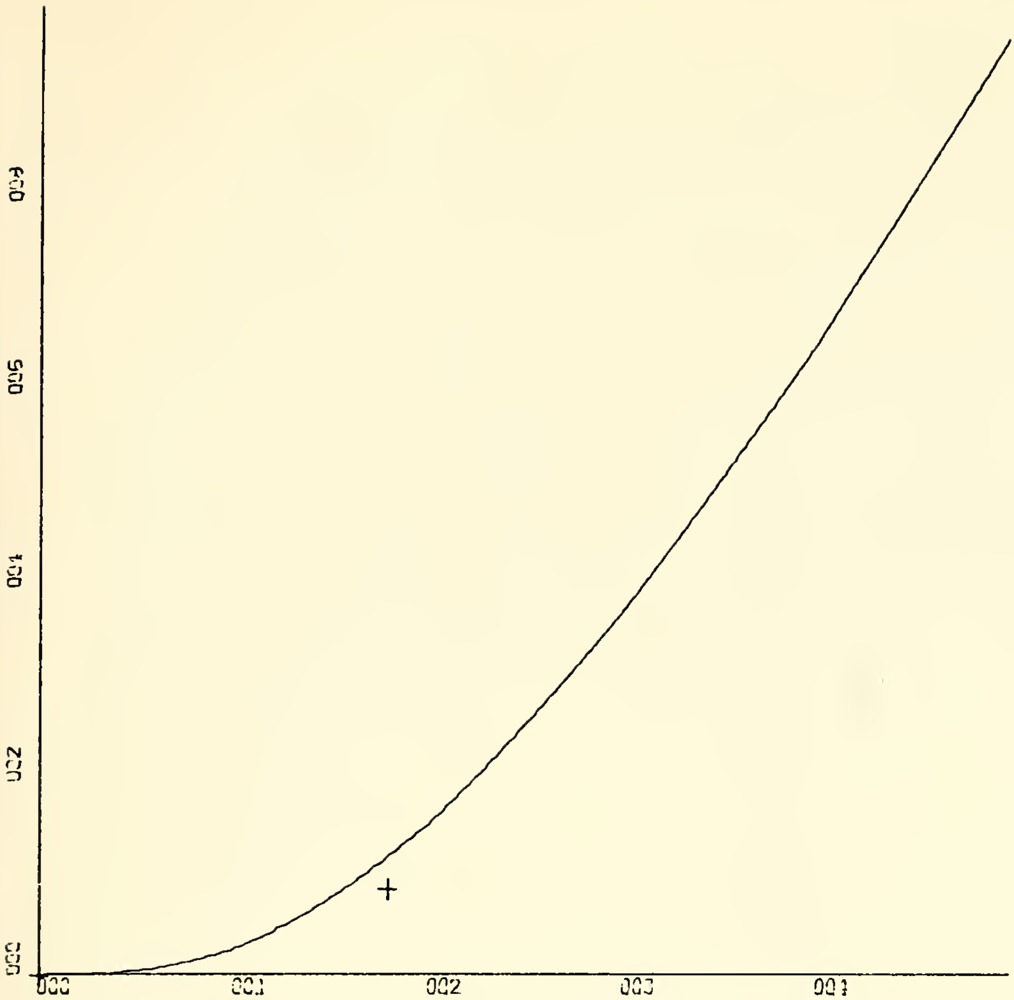
FIGURE 100



FOR DELTA T = 1.6 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01138 |
| 1.6 | 0.67862 |
| 3.2 | 3.92797 |

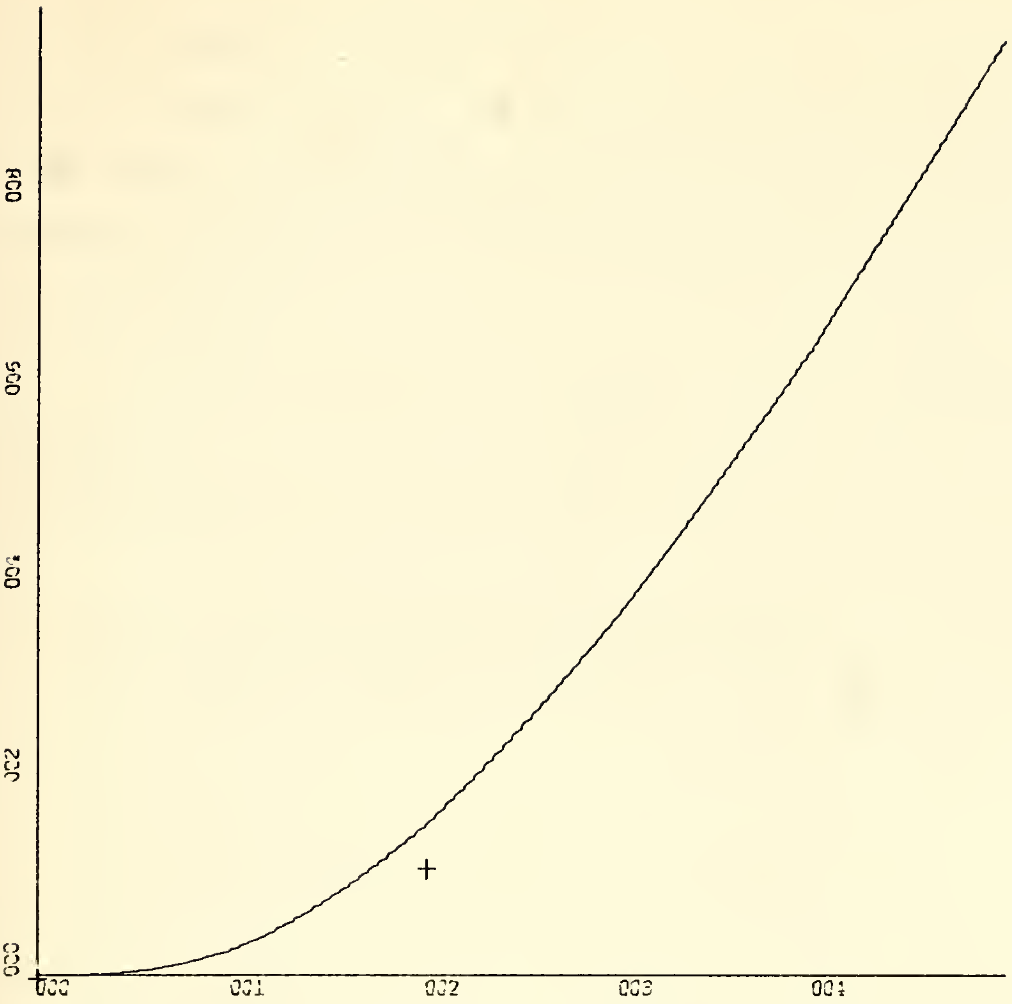
FIGURE 101



FOR DELTA T = 1.8   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01620 |
| 1.8 | 0.88064 |

FIGURE 102



FOR DELTA T = 2.0   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.02222 |
| 2.0 | 1.10457 |

FIGURE 103

1. Case 8 – $\dot{Y} + tY = t^2$

Same calculations as in case 7 with exception for the substitution on the value of P during the division steps, of Equation (47), repeated here, which now has the value P = t.

The algorithm for computer solution (Program 8, page 144), the output for several time iterations and the respective graphs follow (Figs. 104 to 117)

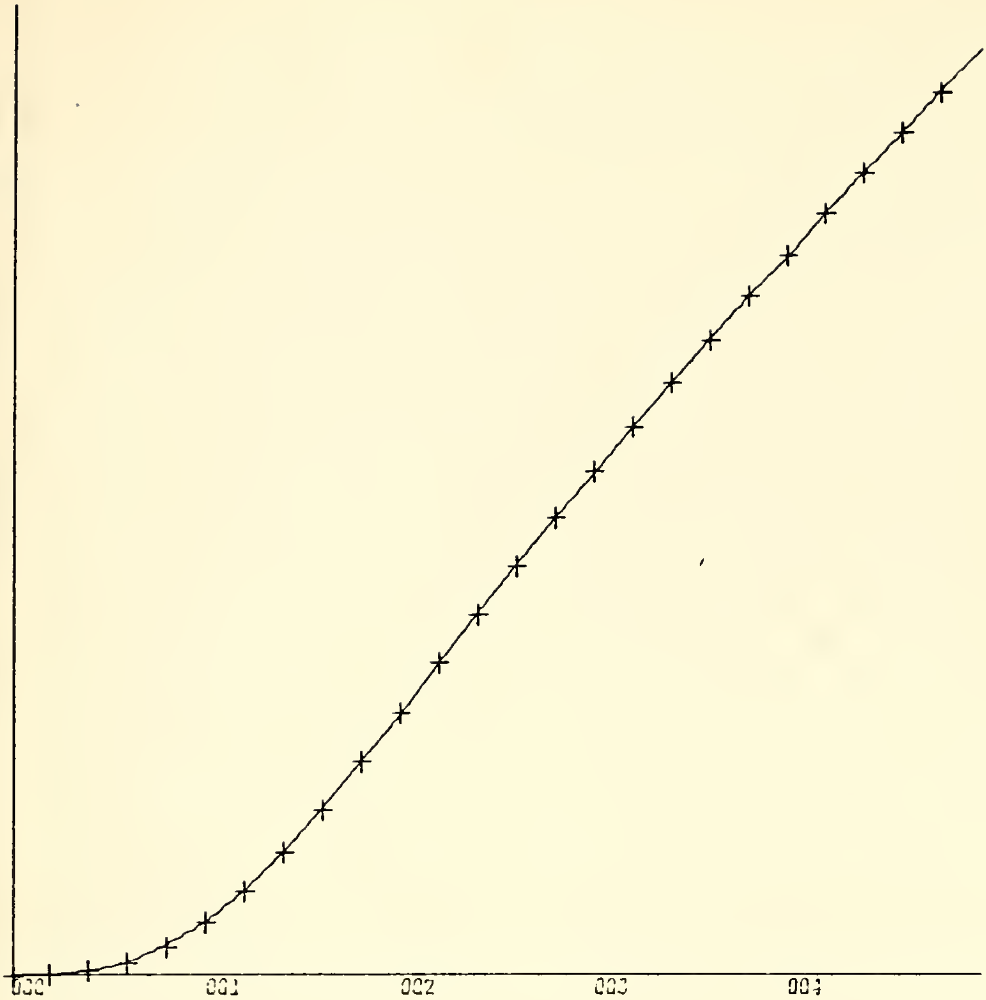
$$y_A^*(z) = \frac{-T^3+124T^3)z^{-1}+(474T^3z^{-2}+124T^3z^{-3}-T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+\ldots \\ \ldots+(360-180PT)z^{-4}}$$

FOR DELTA T = 0.2 TFIN = 5.0

| T | Q(N) | PR |
|---|---|---|
| 0.0 | -0.00002 | 0.0 |
| 0.2 | 0.00261 | 0.00264 |
| 0.4 | 0.02041 | 0.02066 |
| 0.6 | 0.06629 | 0.06707 |
| 0.8 | 0.14905 | 0.15068 |
| 1.0 | 0.27254 | 0.27522 |
| 1.2 | 0.43567 | 0.43942 |
| 1.4 | 0.63339 | 0.63801 |
| 1.6 | 0.85809 | 0.86328 |
| 1.8 | 1.10124 | 1.10660 |
| 2.0 | 1.35473 | 1.35990 |
| 2.2 | 1.61184 | 1.61660 |
| 2.4 | 1.86765 | 1.87190 |
| 2.6 | 2.11908 | 2.12280 |
| 2.8 | 2.36463 | 2.36780 |
| 3.0 | 2.60393 | 2.60670 |
| 3.2 | 2.83735 | 2.83980 |
| 3.4 | 3.06563 | 3.06780 |
| 3.6 | 3.28958 | 3.29160 |
| 3.8 | 3.51003 | 3.51190 |
| 4.0 | 3.72765 | 3.72940 |
| 4.2 | 3.94301 | 3.94460 |
| 4.4 | 4.15655 | 4.15810 |
| 4.6 | 4.36861 | 4.37000 |
| 4.8 | 4.57945 | 4.58080 |

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C  Q(T) = T**2  &  P(T) =T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = T
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT (' T=',8X,' Q(N)=')
    8 FORMAT ('  XX=',7X,'  PR=')
      END
```

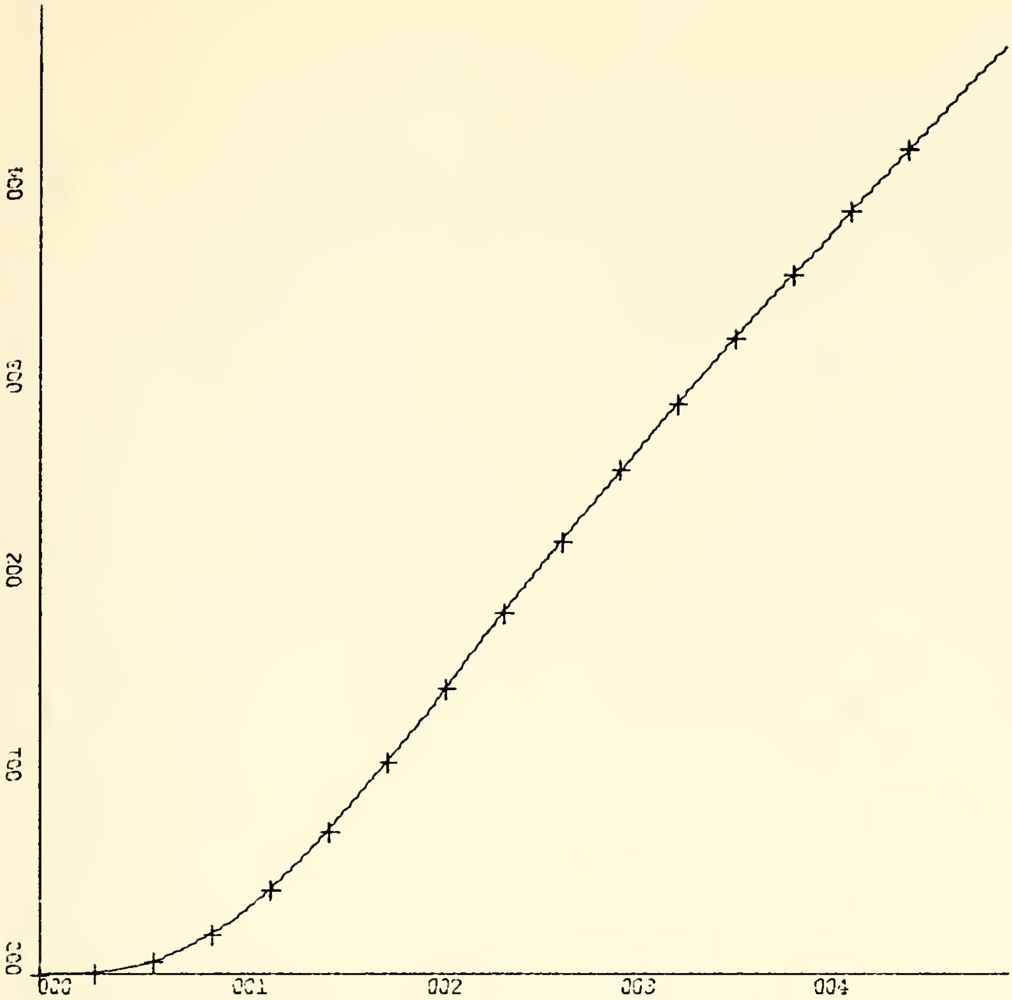
PROGRAM 8



x-scale = 1.0 units/inch
y-scale = 1.0 units/inch

FOR DELTA T = 0.2

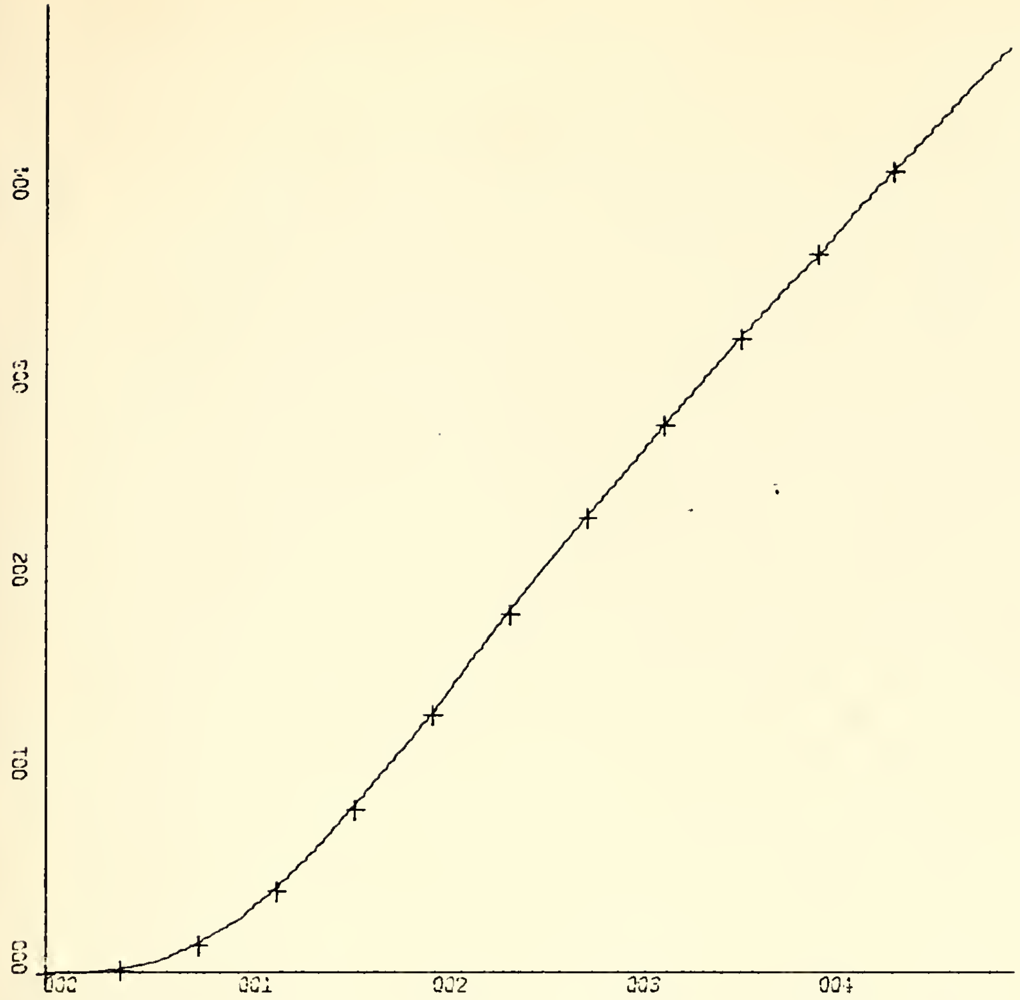
FIGURE 104



FOR DELTA T = 0.3 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00007 |
| 0.3 | 0.00861 |
| 0.6 | 0.06534 |
| 0.9 | 0.20305 |
| 1.2 | 0.43105 |
| 1.5 | 0.73670 |
| 1.8 | 1.09444 |
| 2.1 | 1.47676 |
| 2.4 | 1.86218 |
| 2.7 | 2.23828 |
| 3.0 | 2.60051 |
| 3.3 | 2.94932 |
| 3.6 | 3.28723 |
| 3.9 | 3.61706 |
| 4.2 | 3.94109 |
| 4.5 | 4.26094 |

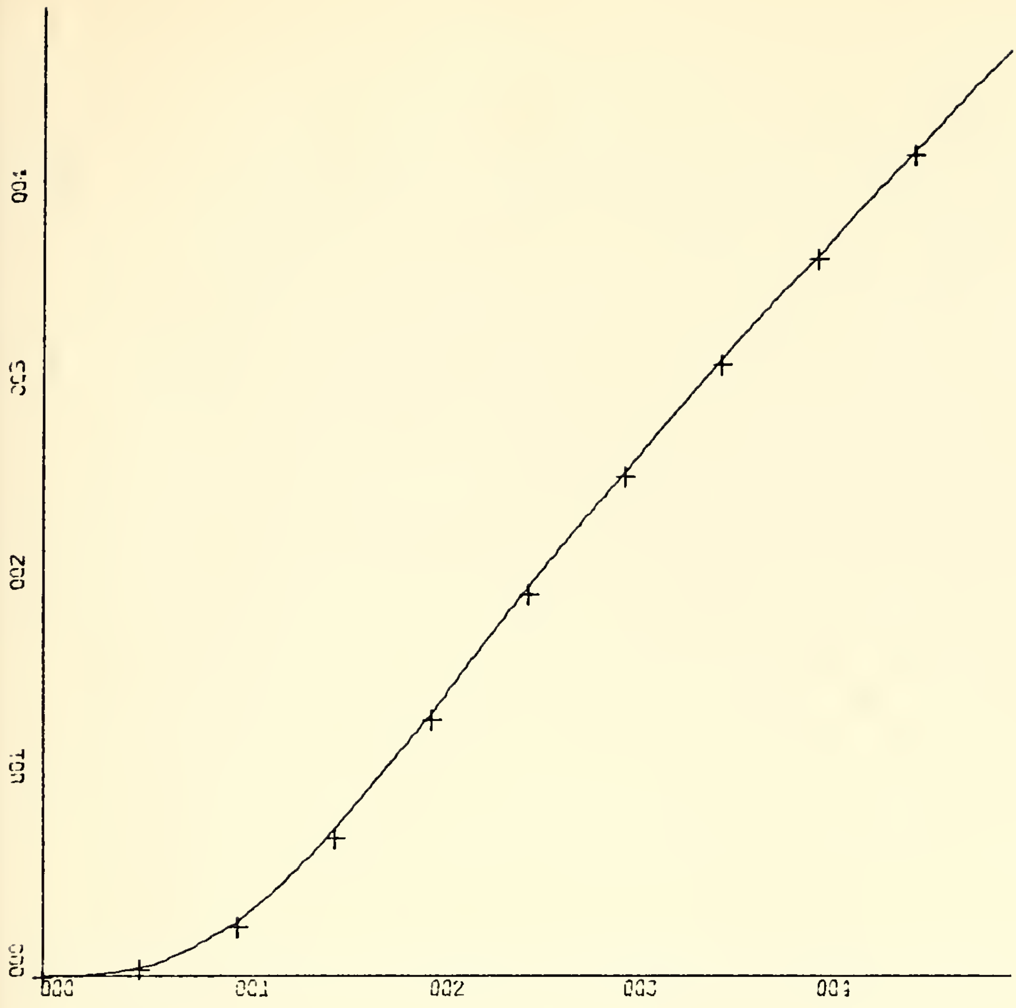
FIGURE 105



FOR DELTA T = 0.4    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00018 |
| 0.4 | 0.01975 |
| 0.8 | 0.14440 |
| 1.2 | 0.42470 |
| 1.6 | 0.84250 |
| 2.0 | 1.33874 |
| 2.4 | 1.85444 |
| 2.8 | 2.35490 |
| 3.2 | 2.83016 |
| 3.6 | 3.28380 |
| 4.0 | 3.72261 |
| 4.4 | 4.15198 |

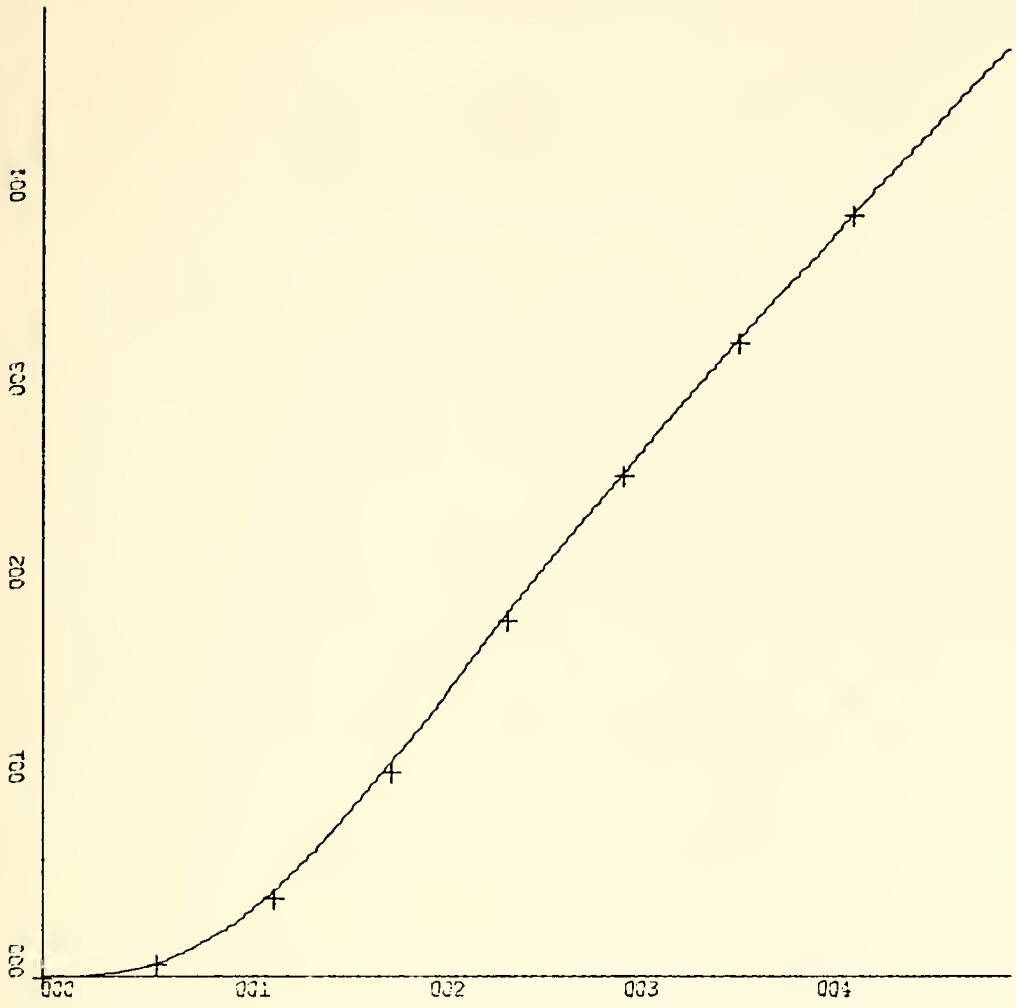
FIGURE 106



FOR DELTA T = 0.5 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00035 |
| 0.5 | 0.03704 |
| 1.0 | 0.25926 |
| 1.5 | 0.71717 |
| 2.0 | 1.32660 |
| 2.5 | 1.97228 |
| 3.0 | 2.58929 |
| 3.5 | 3.16744 |
| 4.0 | 3.71877 |
| 4.5 | 4.25487 |

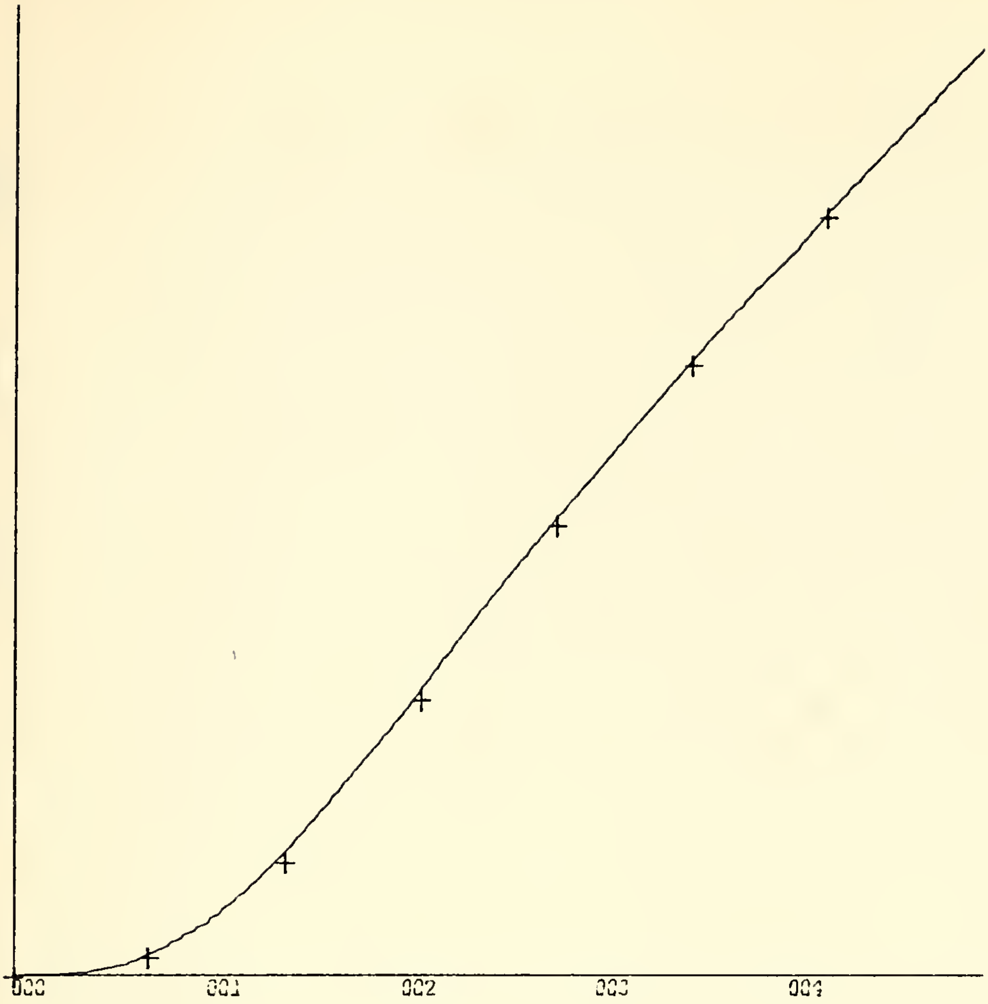
FIGURE 107



FOR DELTA T = 0.6    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00060 |
| 0.6 | 0.06102 |
| 1.2 | 0.40738 |
| 1.8 | 1.05761 |
| 2.4 | 1.83168 |
| 3.0 | 2.58151 |
| 3.6 | 3.27410 |
| 4.2 | 3.93011 |

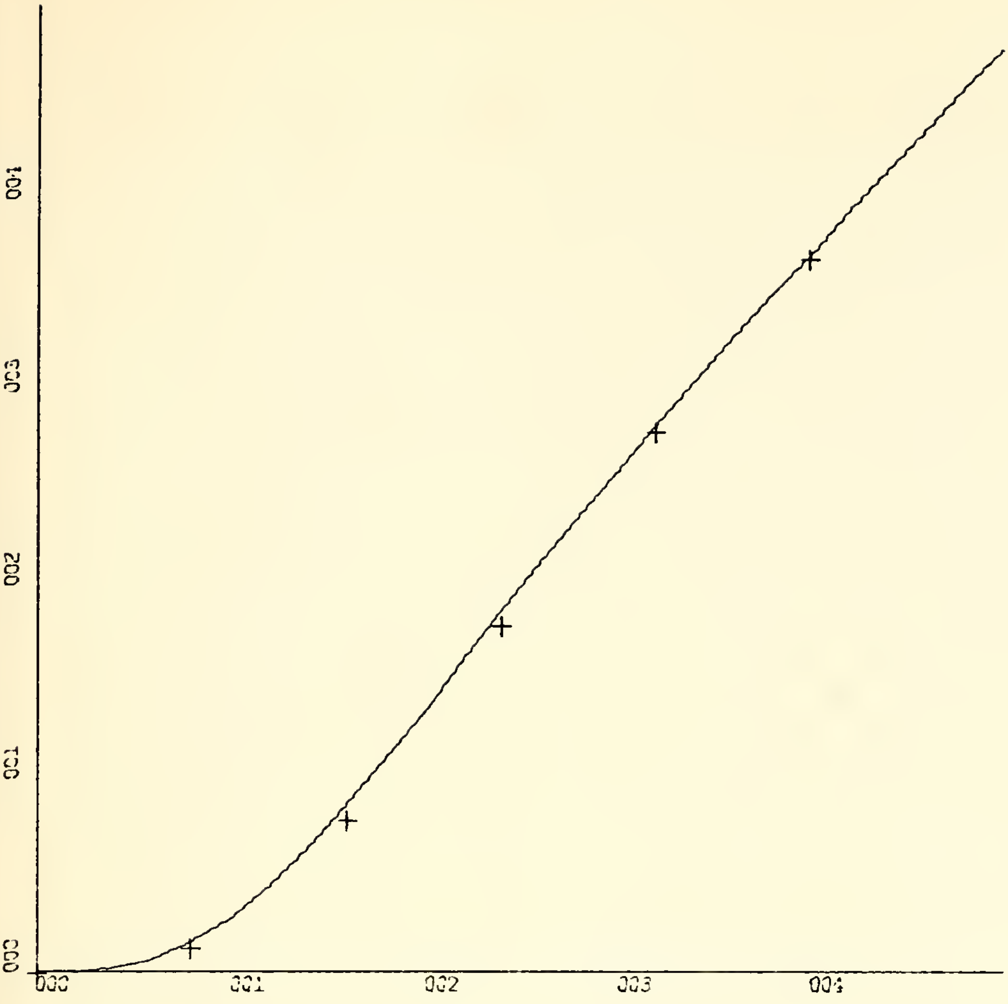
FIGURE 108



FOR DELTA T = 0.7 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00095 |
| 0.7 | 0.09183 |
| 1.4 | 0.58367 |
| 2.1 | 1.42363 |
| 2.8 | 2.32707 |
| 3.5 | 3.15544 |
| 4.2 | 3.92482 |

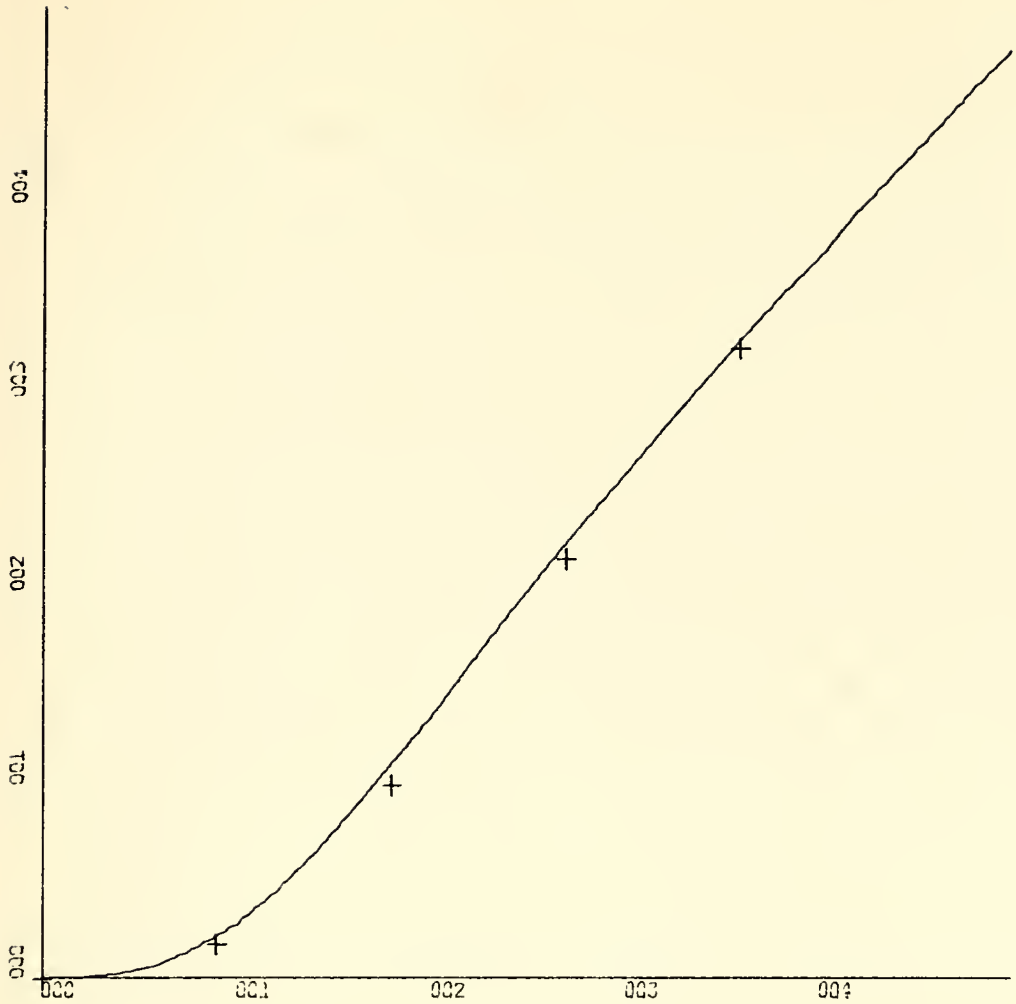
FIGURE 109



FOR DELTA T = 0.8 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00142 |
| 0.8 | 0.12929 |
| 1.6 | 0.78206 |
| 2.4 | 1.79807 |
| 3.2 | 2.80113 |
| 4.0 | 3.70243 |

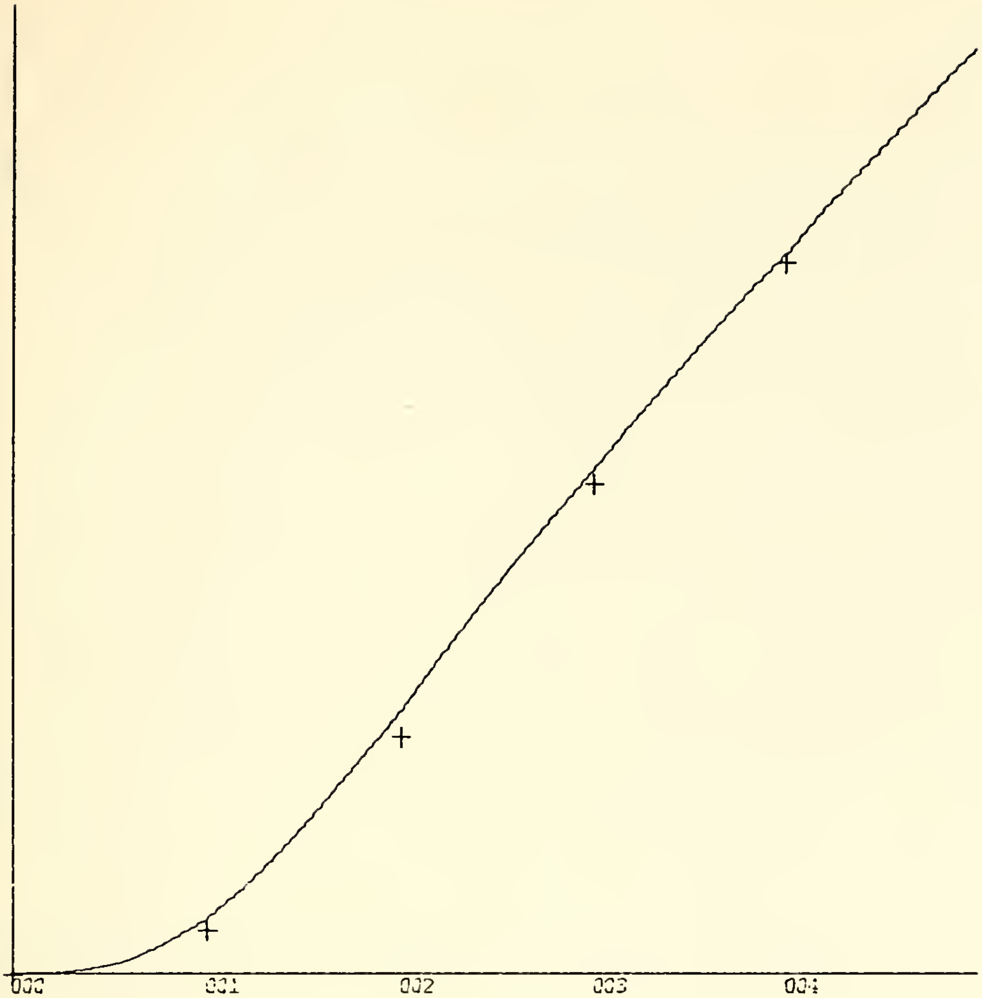
FIGURE 110



FOR DELTA T = 0.9   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | =0.00202 |
| 0.9 | 0.17295 |
| 1.8 | 0.99663 |
| 2.7 | 2.16991 |
| 3.6 | 3.25361 |

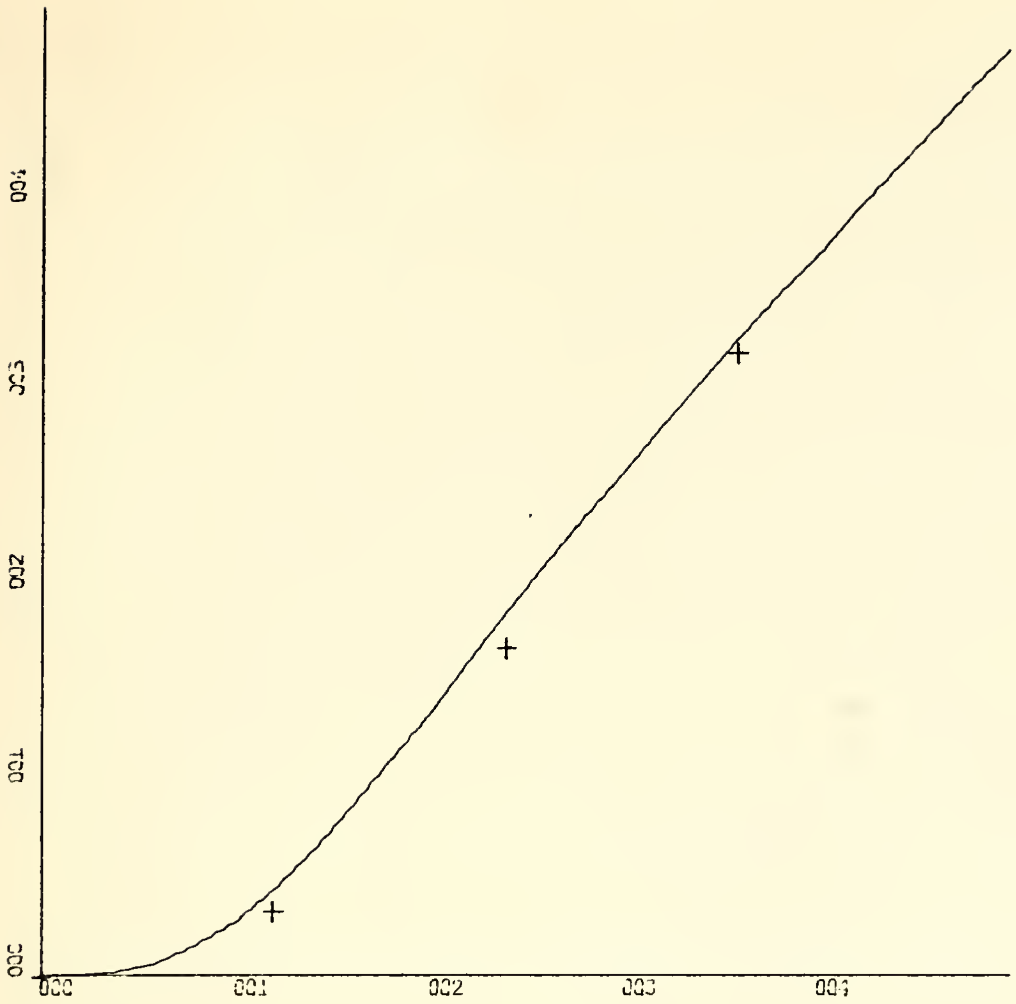
FIGURE 111



FOR DELTA T = 1.0    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00278 |
| 1.0 | 0.22222 |
| 2.0 | 1.22222 |
| 3.0 | 2.53333 |
| 4.0 | 3.68888 |

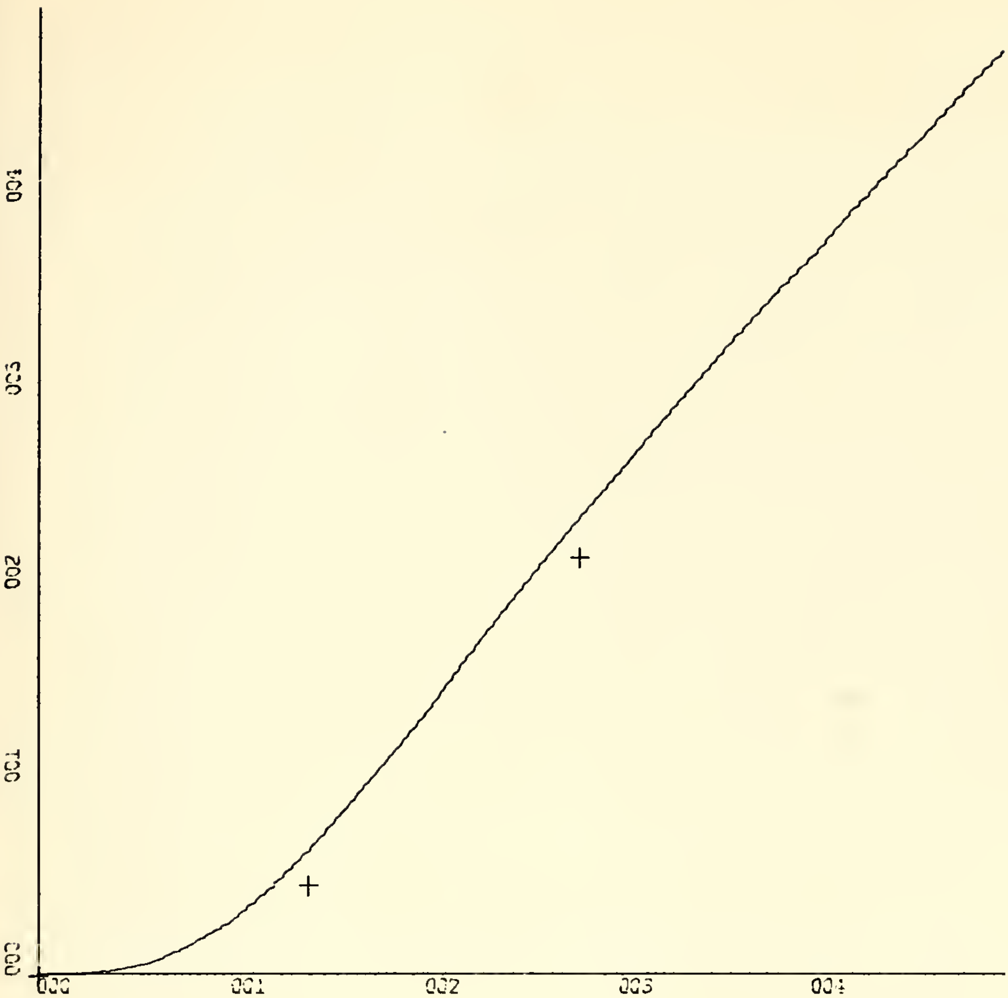
FIGURE 112



FOR DELTA T = 1.2   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00480 |
| 1.2 | 0.33488 |
| 2.4 | 1.69089 |
| 3.6 | 3.22784 |

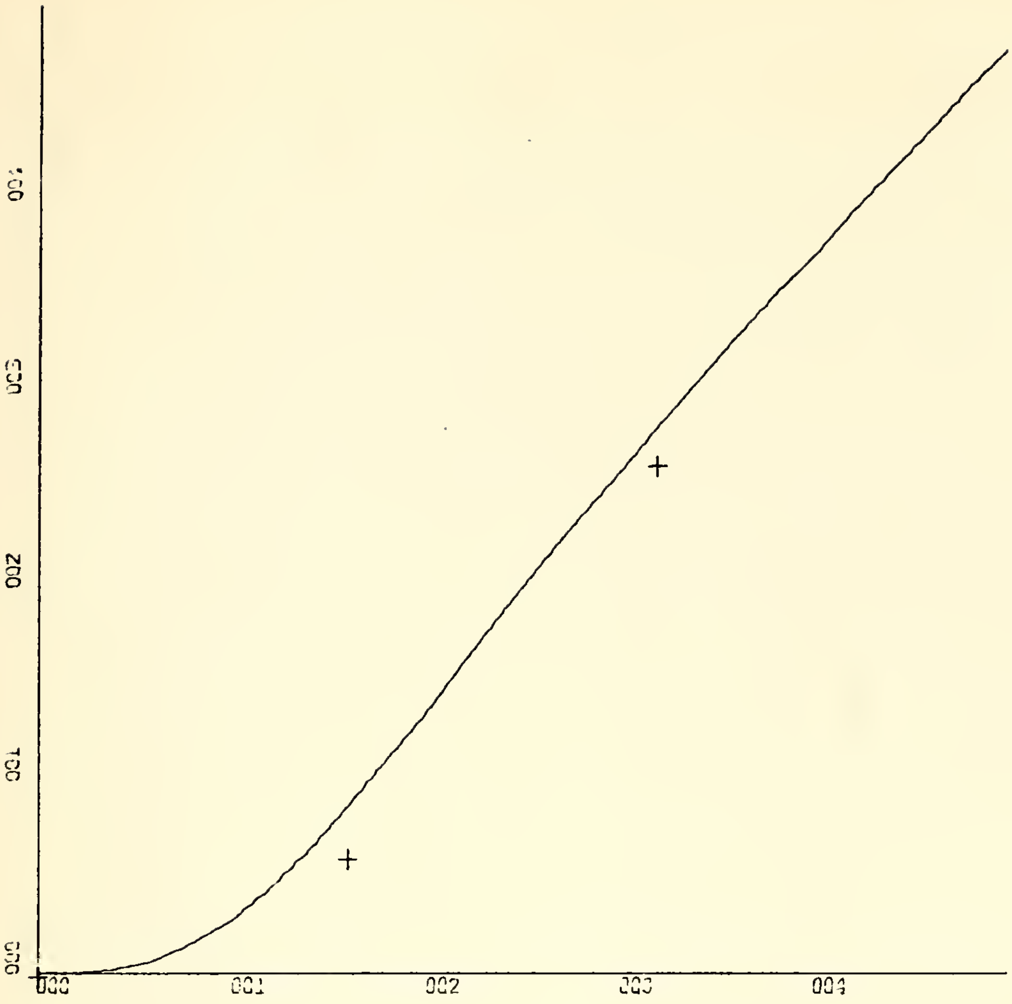
FIGURE 113



FOR DELTA T = 1.4 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00762 |
| 1.4 | 0.46195 |
| 2.8 | 2.16618 |

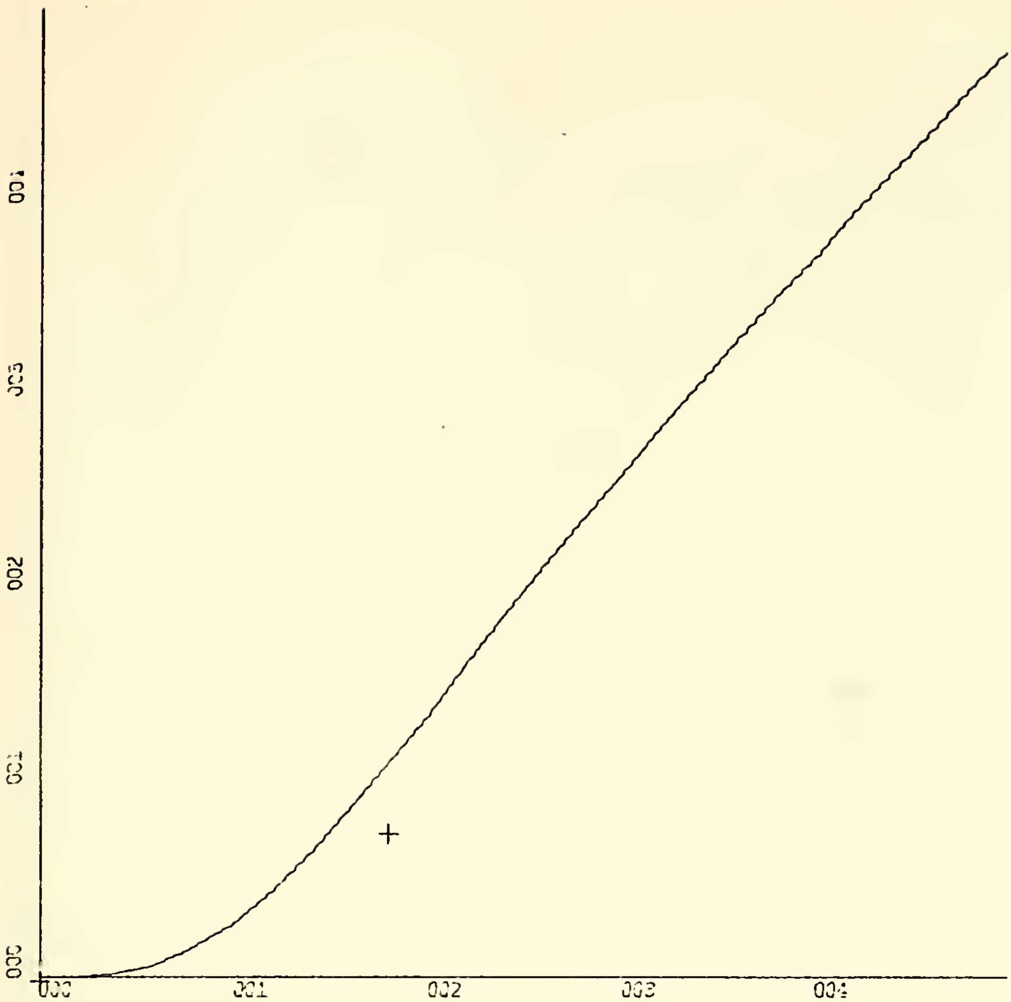
FIGURE 114



FOR DELTA T = 1.6    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01138 |
| 1.6 | 0.59883 |
| 3.2 | 2.63754 |

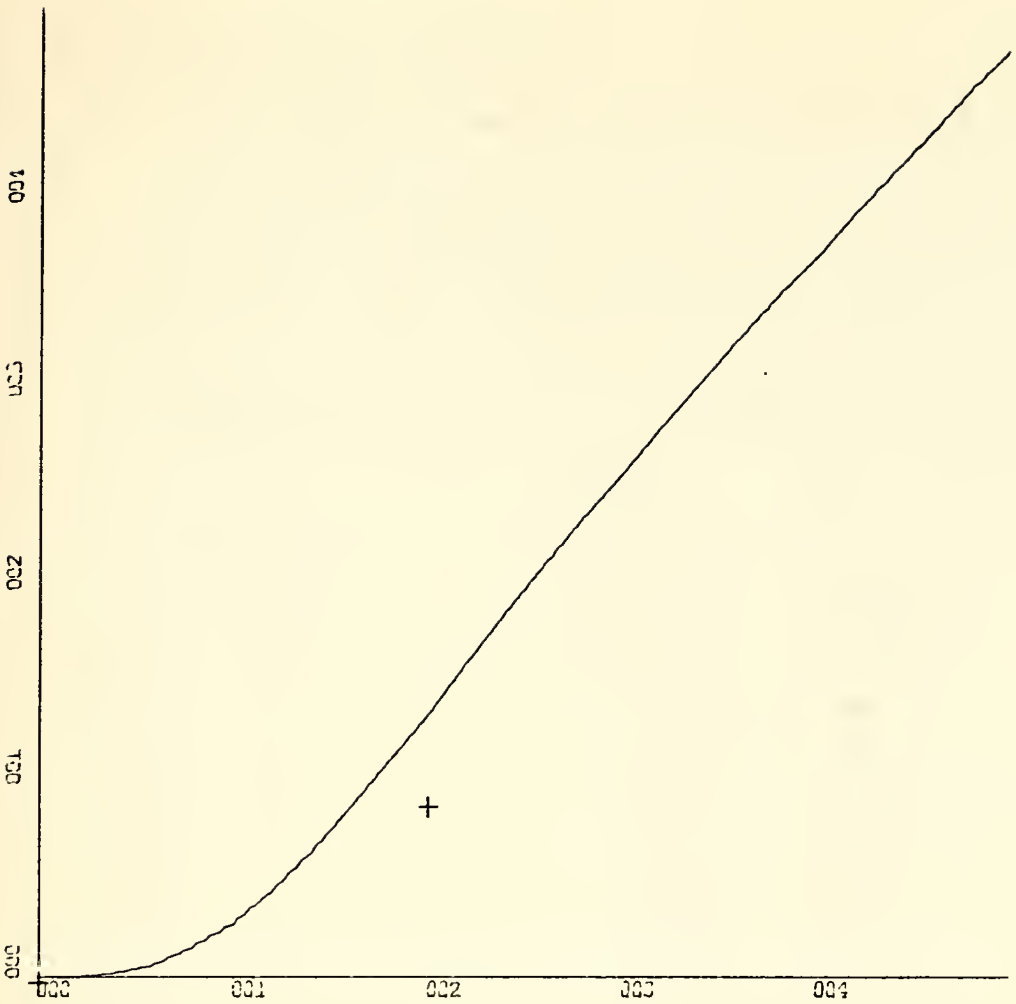
FIGURE 115



FOR DELTA T = 1.8 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01620 |
| 1.8 | 0.74198 |

FIGURE 116



FOR DELTA T = 2.0 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.02222 |
| 2.0 | 0.88889 |

FIGURE 117

2. <u>Case 9 — $\dot{Y} + t^2Y = t^2$</u>

Same reasoning as in case 8. This time $P = t^2$.

Computer algorithm for solution of this problem is shown in Program 9 (Page 160). Solutions and graphs for several T's follow in Figures 118 to 131.

FOR DELTA T = 0.2 TFIN = 5.0

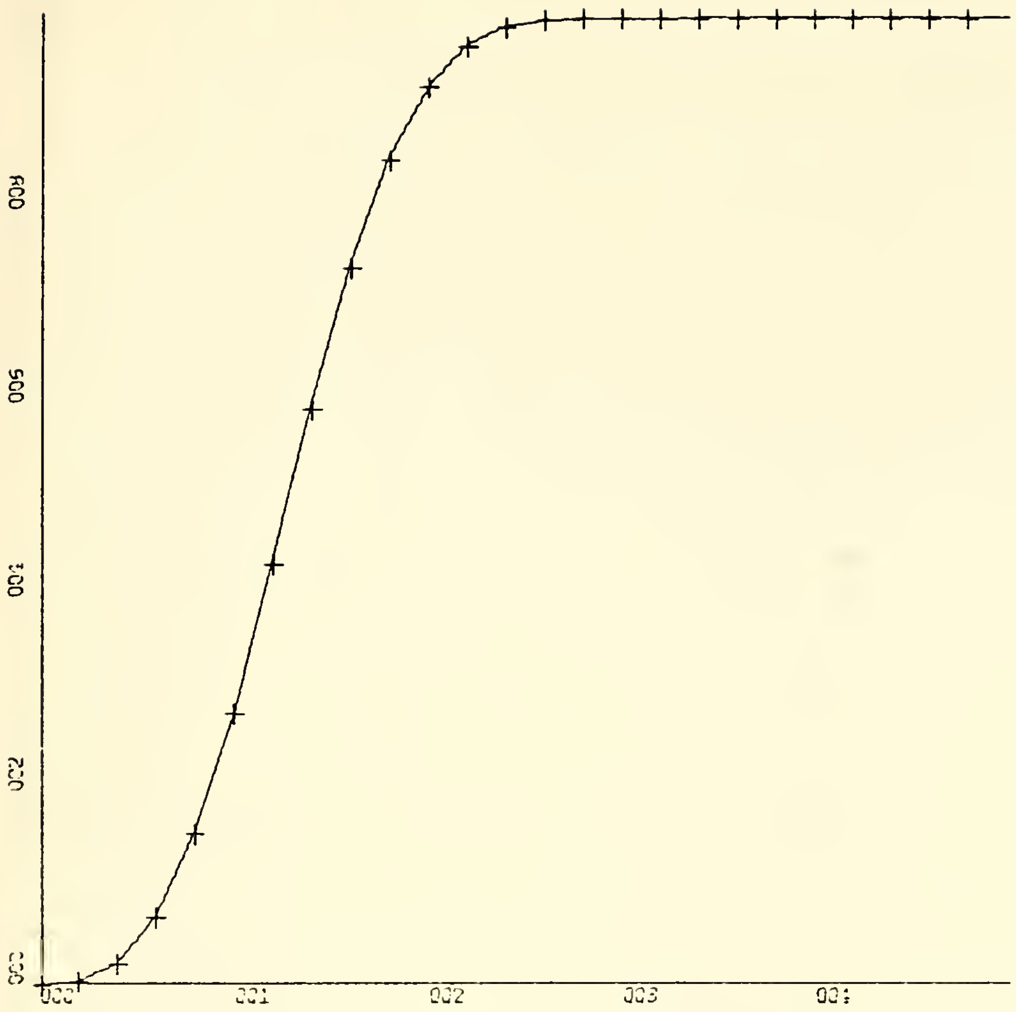
| T | Q(N) | PR |
|---|---|---|
| 0.0 | -0.00002 | 0.0 |
| 0.2 | 0.00266 | 0.00266 |
| 0.4 | 0.02098 | 0.02110 |
| 0.6 | 0.06883 | 0.06947 |
| 0.8 | 0.15509 | 0.15690 |
| 1.0 | 0.27985 | 0.28347 |
| 1.2 | 0.43228 | 0.43785 |
| 1.4 | 0.59256 | 0.59934 |
| 1.6 | 0.73812 | 0.74469 |
| 1.8 | 0.85183 | 0.85686 |
| 2.0 | 0.92749 | 0.93051 |
| 2.2 | 0.96977 | 0.97125 |
| 2.4 | 0.98924 | 0.99002 |
| 2.6 | 0.99645 | 0.99714 |
| 2.8 | 0.99857 | 0.99934 |
| 3.0 | 0.99909 | 0.99988 |
| 3.2 | 0.99924 | 0.99998 |
| 3.4 | 0.99932 | 1.00000 |
| 3.6 | 0.99938 | 1.00000 |
| 3.8 | 0.99943 | 1.00000 |
| 4.0 | 0.99947 | 1.00000 |
| 4.2 | 0.99950 | 1.00000 |
| 4.4 | 0.99952 | 1.00000 |
| 4.6 | 0.99954 | 1.00000 |
| 4.8 | 0.99956 | 1.00000 |

Equation (47) is repeated for convenience:

$$y_A^*(z) = \frac{-T^3+(124T^3)z^{-1}+(474T^3)z^{-2}+124T^3z^{-3}-T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+\dots \dots+(360-180PT)z^{-4}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T) = T**2  &   P(T) =T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = T**2
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT (' T=',8X,' Q(N)=')
    8 FORMAT ('   XX=',7X,'   PR=')
      END
```

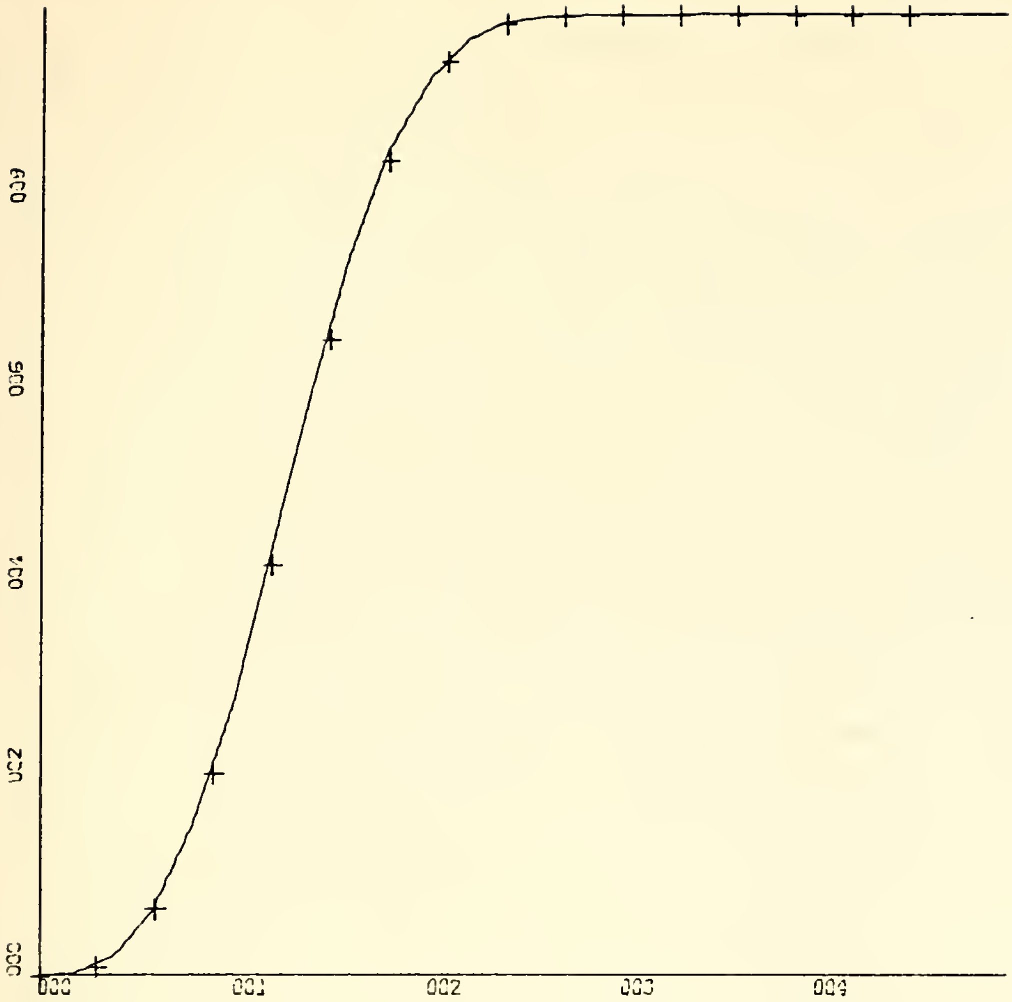
PROGRAM 9



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

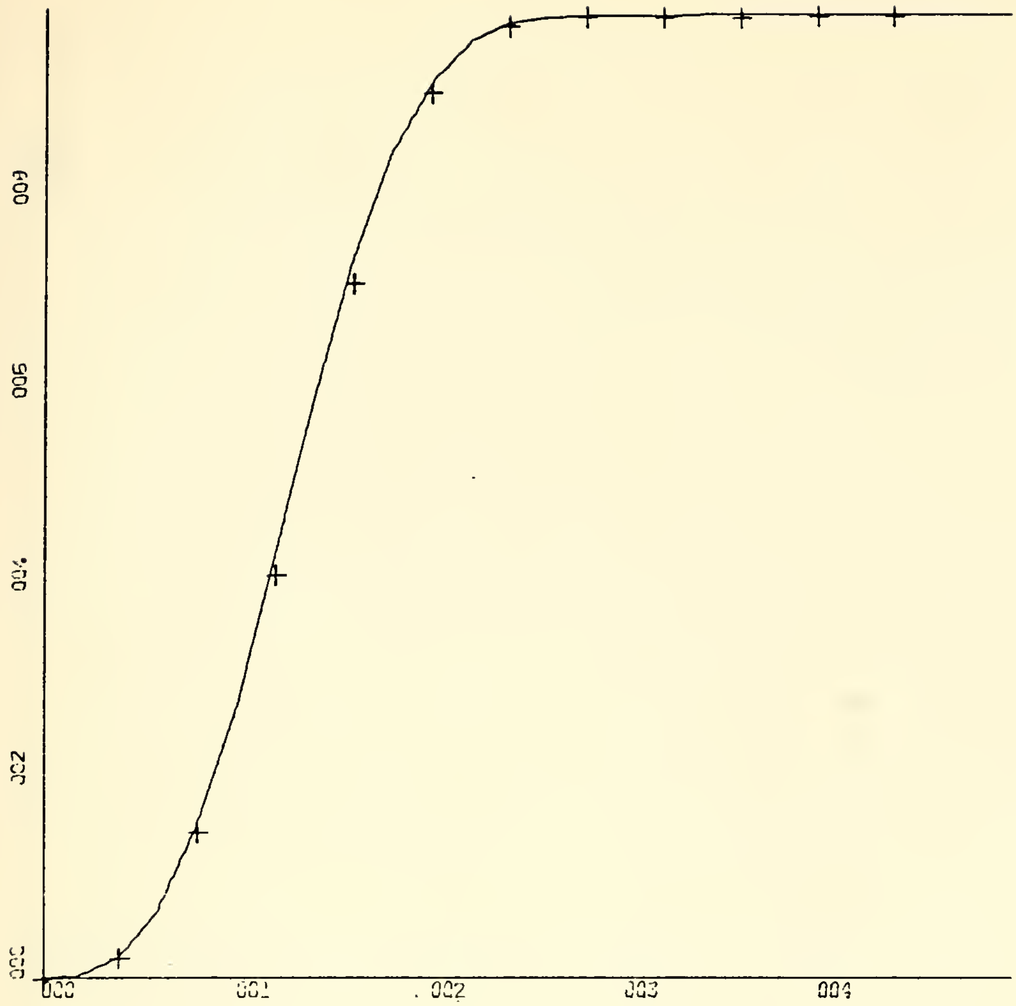
FIGURE 118



FOR DELTA T = 0.3 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00007 |
| 0.3 | 0.00888 |
| 0.6 | 0.06808 |
| 0.9 | 0.20990 |
| 1.2 | 0.42549 |
| 1.5 | 0.65988 |
| 1.8 | 0.84534 |
| 2.1 | 0.94944 |
| 2.4 | 0.98840 |
| 2.7 | 0.99809 |
| 3.0 | 0.99819 |
| 3.3 | 0.99851 |
| 3.6 | 0.99877 |
| 3.9 | 0.99895 |
| 4.2 | 0.99910 |
| 4.5 | 0.99921 |

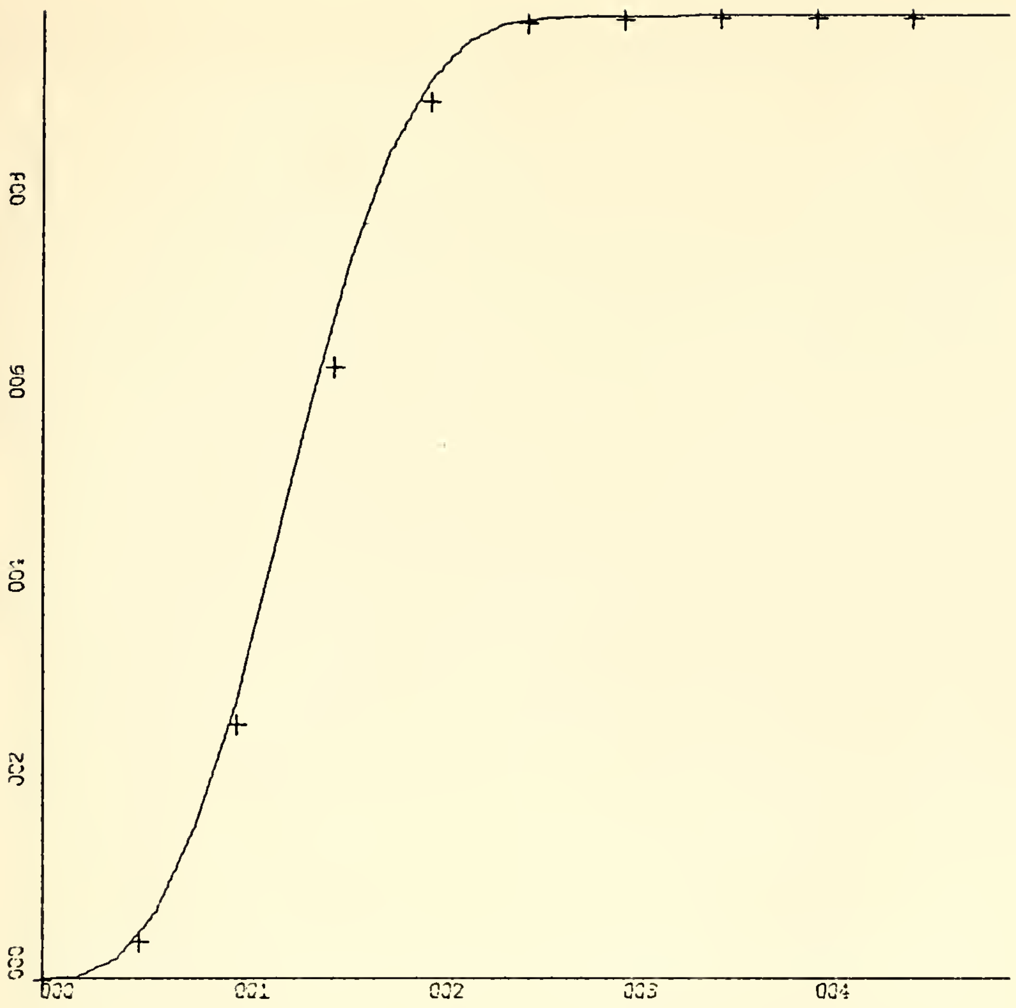
FIGURE 119



FOR DELTA T = 0.4    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00018 |
| 0.4 | 0.02067 |
| 0.8 | 0.15013 |
| 1.2 | 0.41634 |
| 1.6 | 0.71810 |
| 2.0 | 0.91765 |
| 2.4 | 0.98739 |
| 2.8 | 0.99659 |
| 3.2 | 0.99714 |
| 3.6 | 0.99787 |
| 4.0 | 0.99827 |
| 4.4 | 0.99859 |

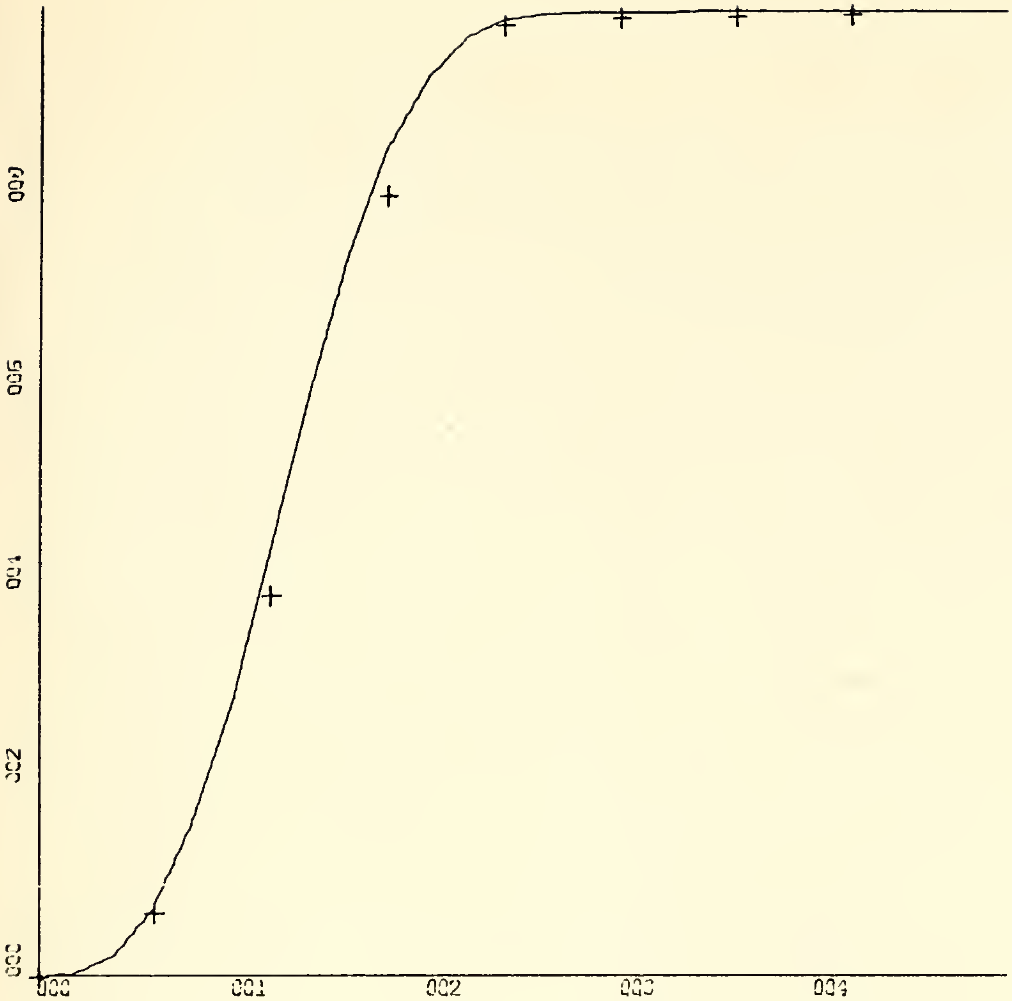
FIGURE 120



FOR DELTA T = 0.5   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00035 |
| 0.5 | 0.03922 |
| 1.0 | 0.26275 |
| 1.5 | 0.63278 |
| 2.0 | 0.90925 |
| 2.5 | 0.99187 |
| 3.0 | 0.99499 |
| 3.5 | 0.99641 |
| 4.0 | 0.99731 |
| 4.5 | 0.99789 |

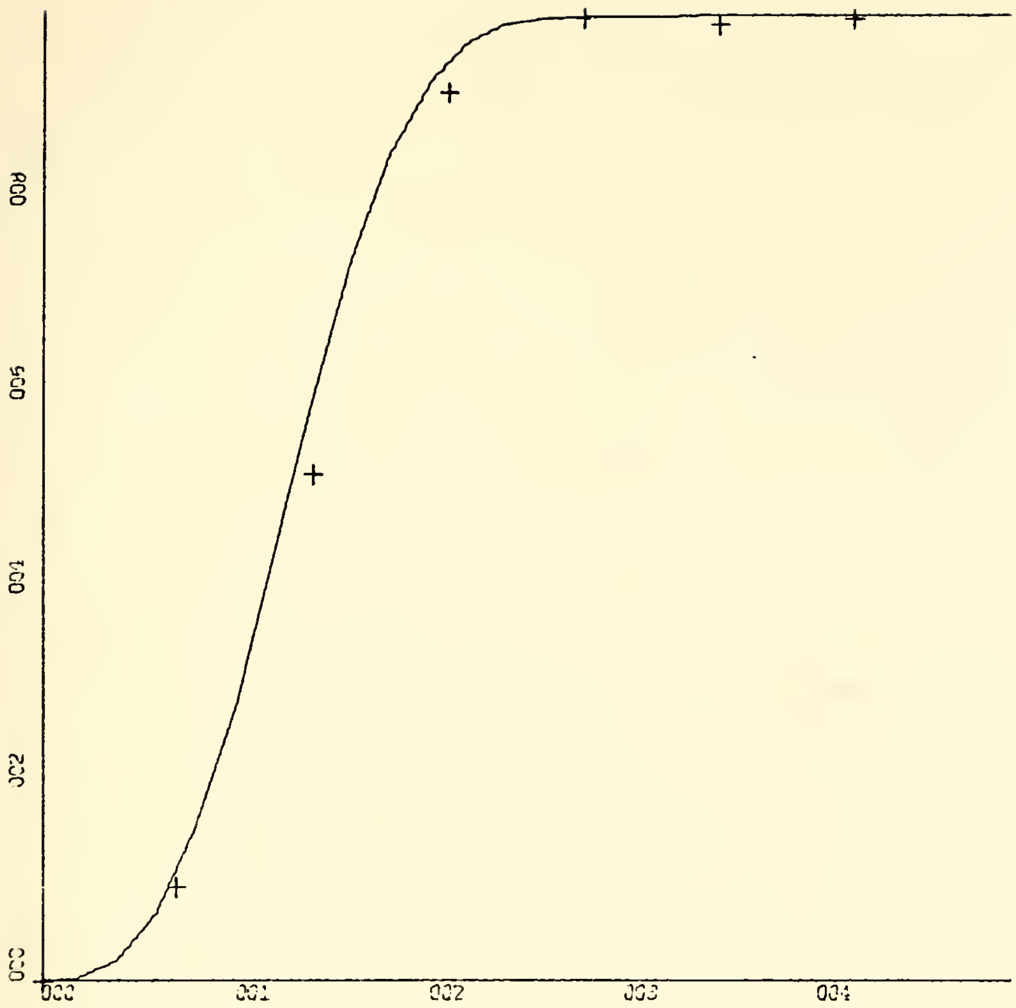
FIGURE 121



FOR DELTA T = 0.6 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00060 |
| 0.6 | 0.06498 |
| 1.2 | 0.39243 |
| 1.8 | 0.80675 |
| 2.4 | 0.98482 |
| 3.0 | 0.99326 |
| 3.6 | 0.99498 |
| 4.2 | 0.99658 |

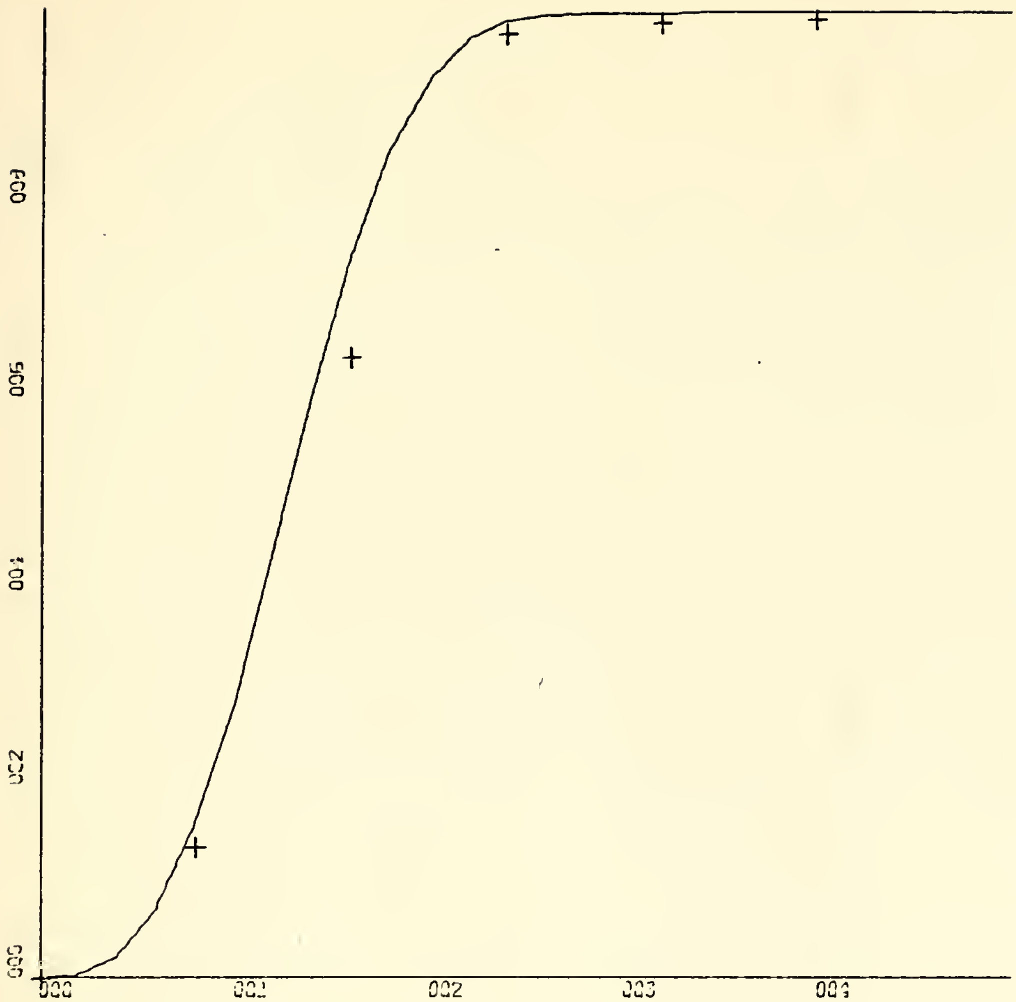
FIGURE 122



FOR DELTA T = 0.7   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00095 |
| 0.7 | 0.09760 |
| 1.4 | 0.52265 |
| 2.1 | 0.91859 |
| 2.8 | 0.99655 |
| 3.5 | 0.99033 |
| 4.2 | 0.99646 |

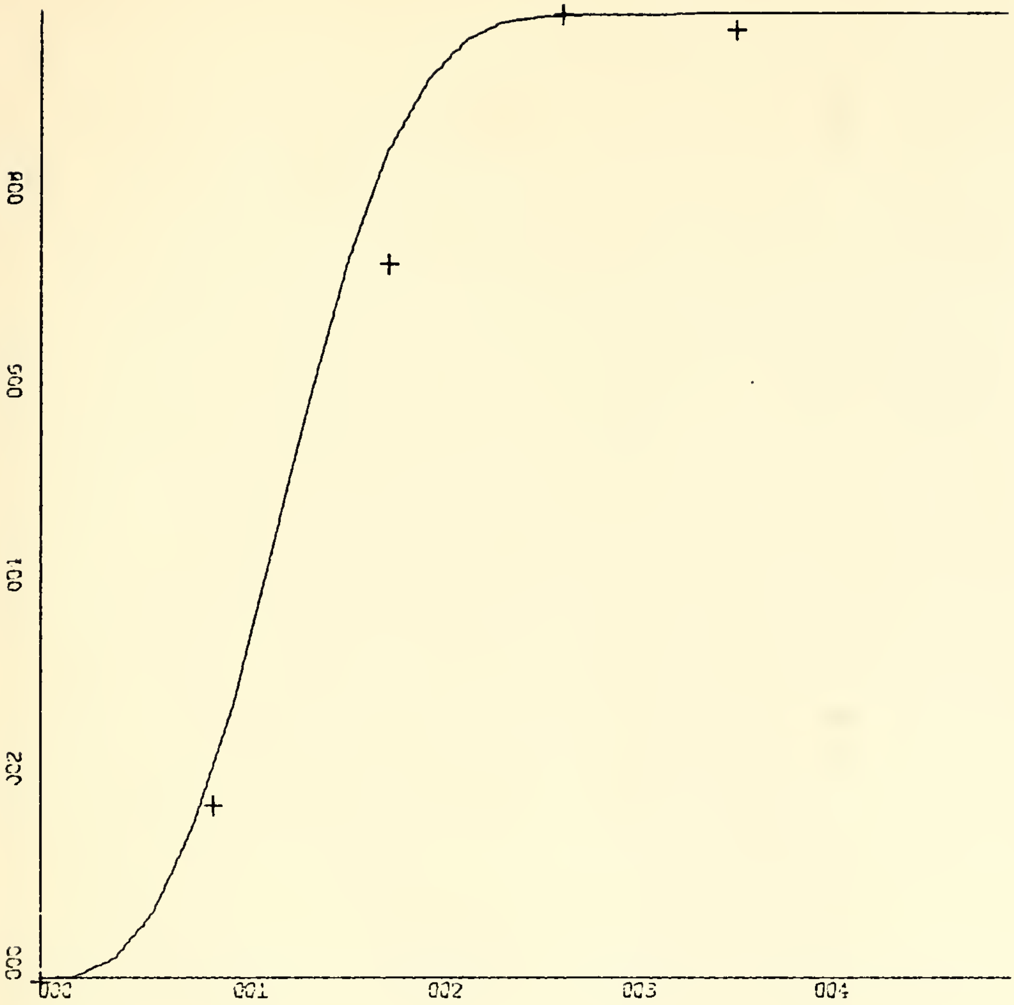
FIGURE 123



FOR DELTA T = 0.8    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00142 |
| 0.8 | 0.13588 |
| 1.6 | 0.64020 |
| 2.4 | 0.97679 |
| 3.2 | 0.98920 |
| 4.0 | 0.99299 |

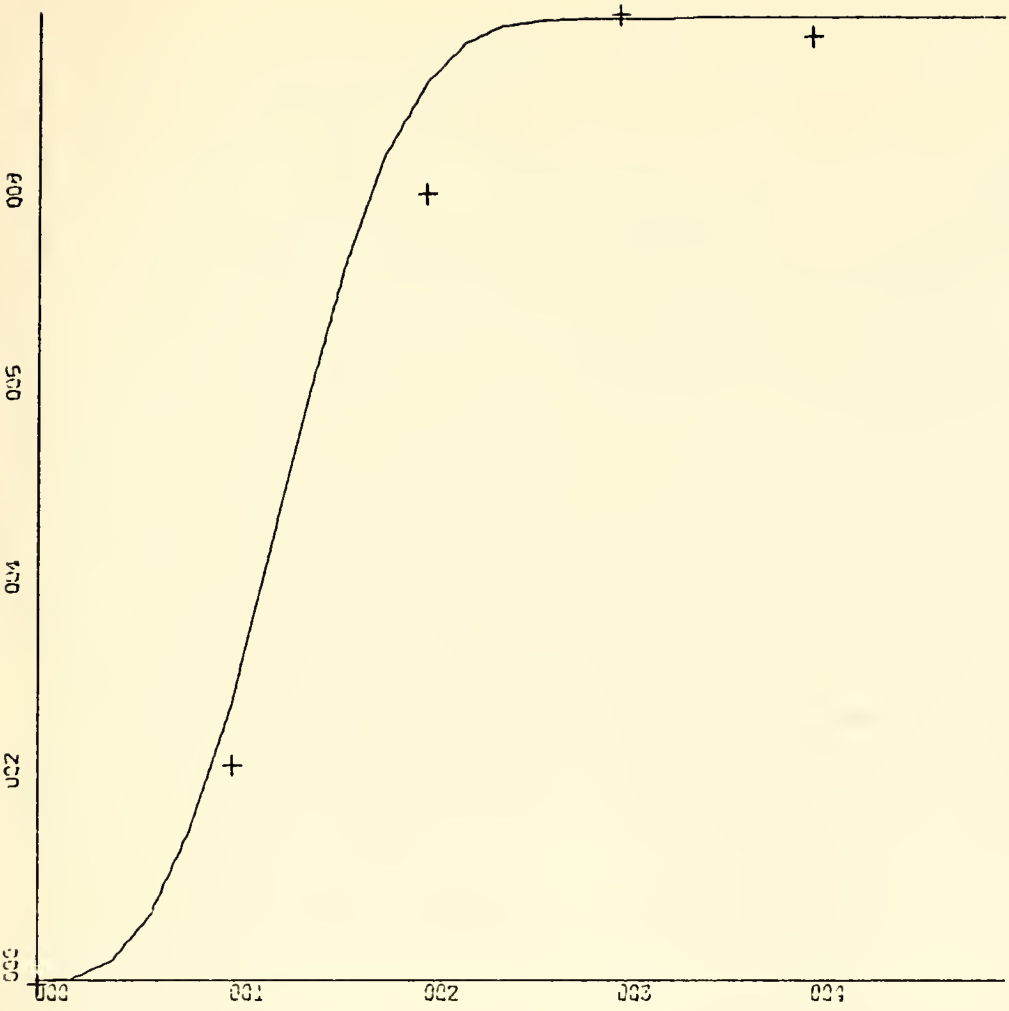
FIGURE 124



FOR DELTA T = 0.9    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00202 |
| 0.9 | 0.17809 |
| 1.8 | 0.73807 |
| 2.7 | 0.99964 |
| 3.6 | 0.98234 |

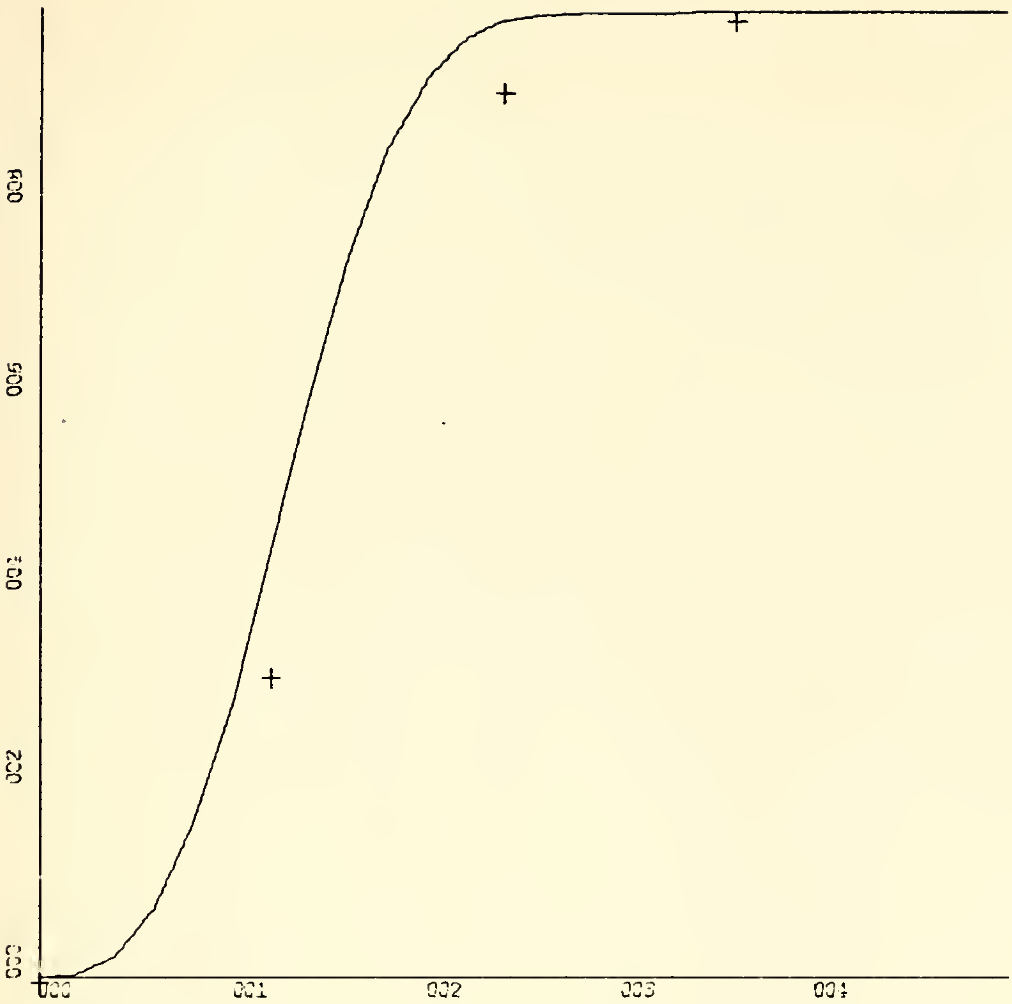
FIGURE 125



FOR DELTA T = 1.0 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00278 |
| 1.0 | 0.22222 |
| 2.0 | 0.81481 |
| 3.0 | 1.00337 |
| 4.0 | 0.98017 |

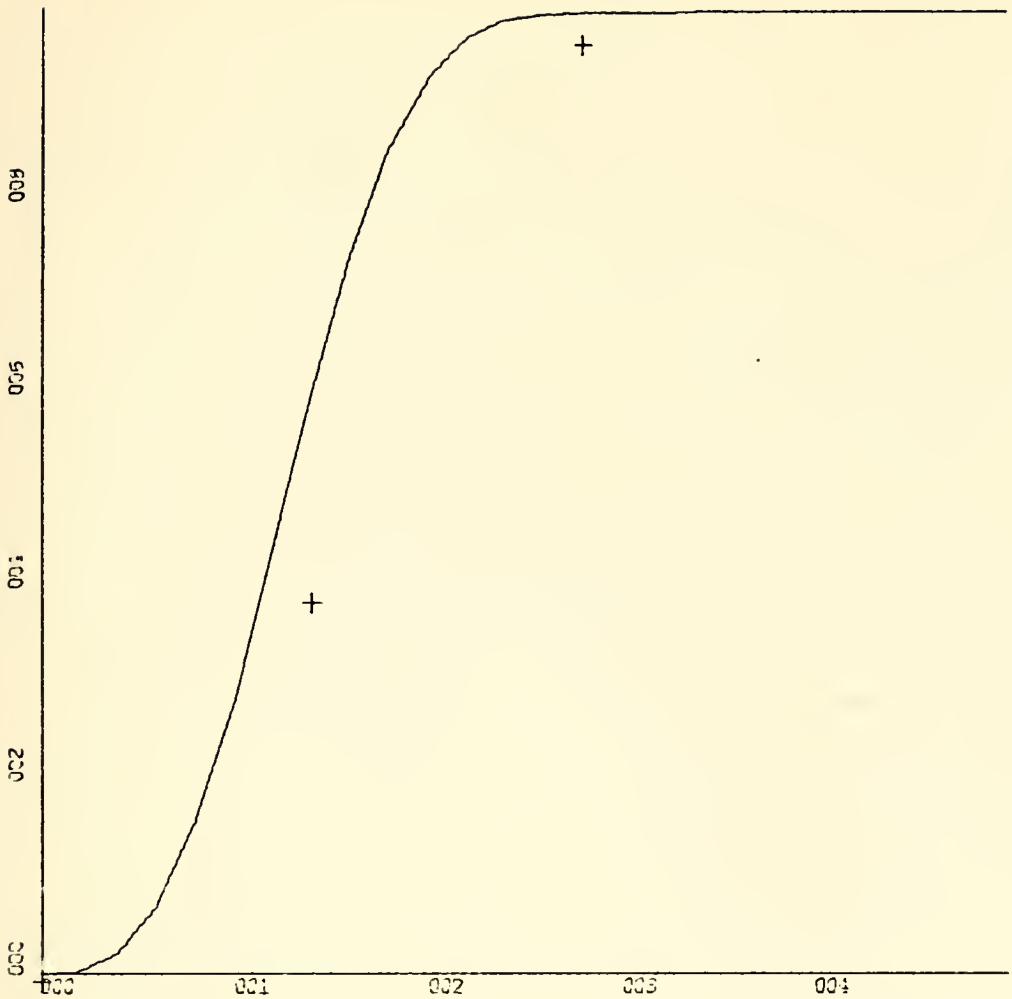
FIGURE 126



FOR DELTA T = 1.2   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00480 |
| 1.2 | 0.30901 |
| 2.4 | 0.91428 |
| 3.6 | 0.99117 |

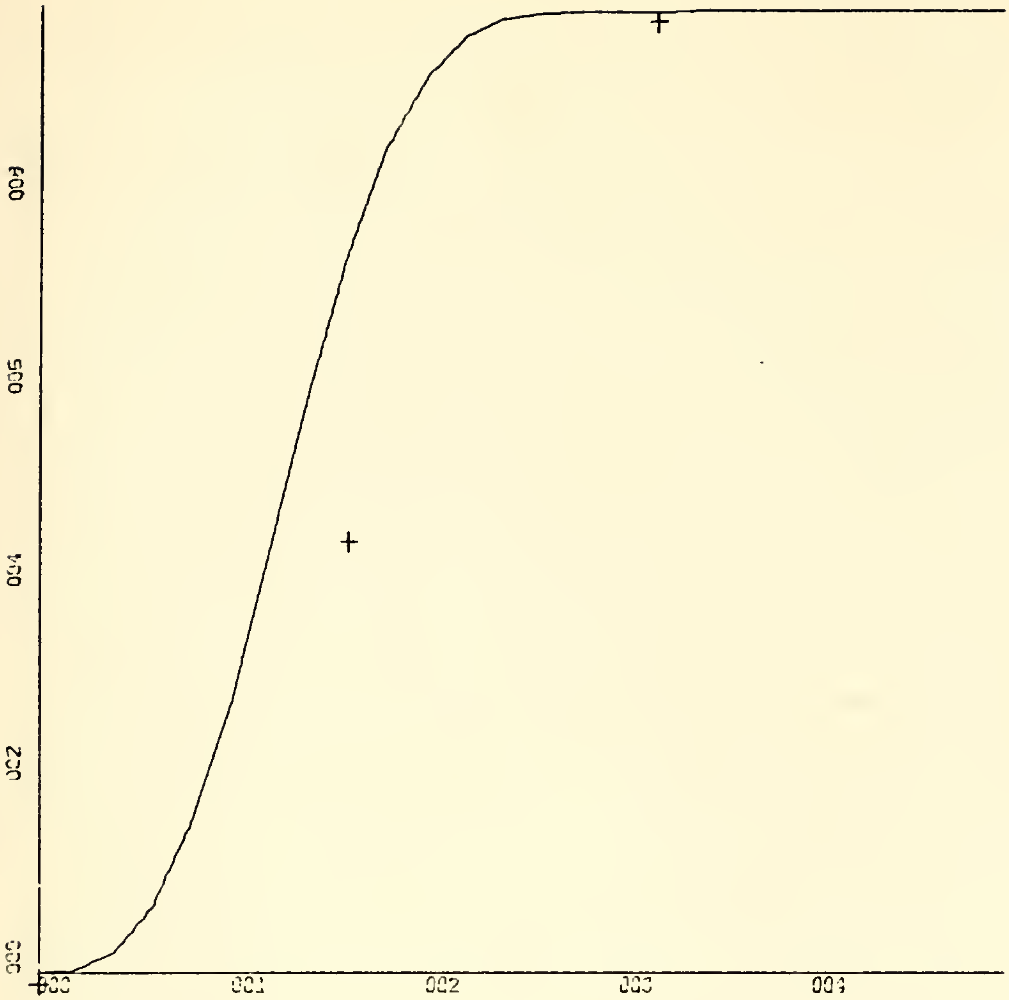
FIGURE 127



FOR DELTA T = 1.4 TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.00762 |
| 1.4 | 0.38561 |
| 2.8 | 0.96474 |

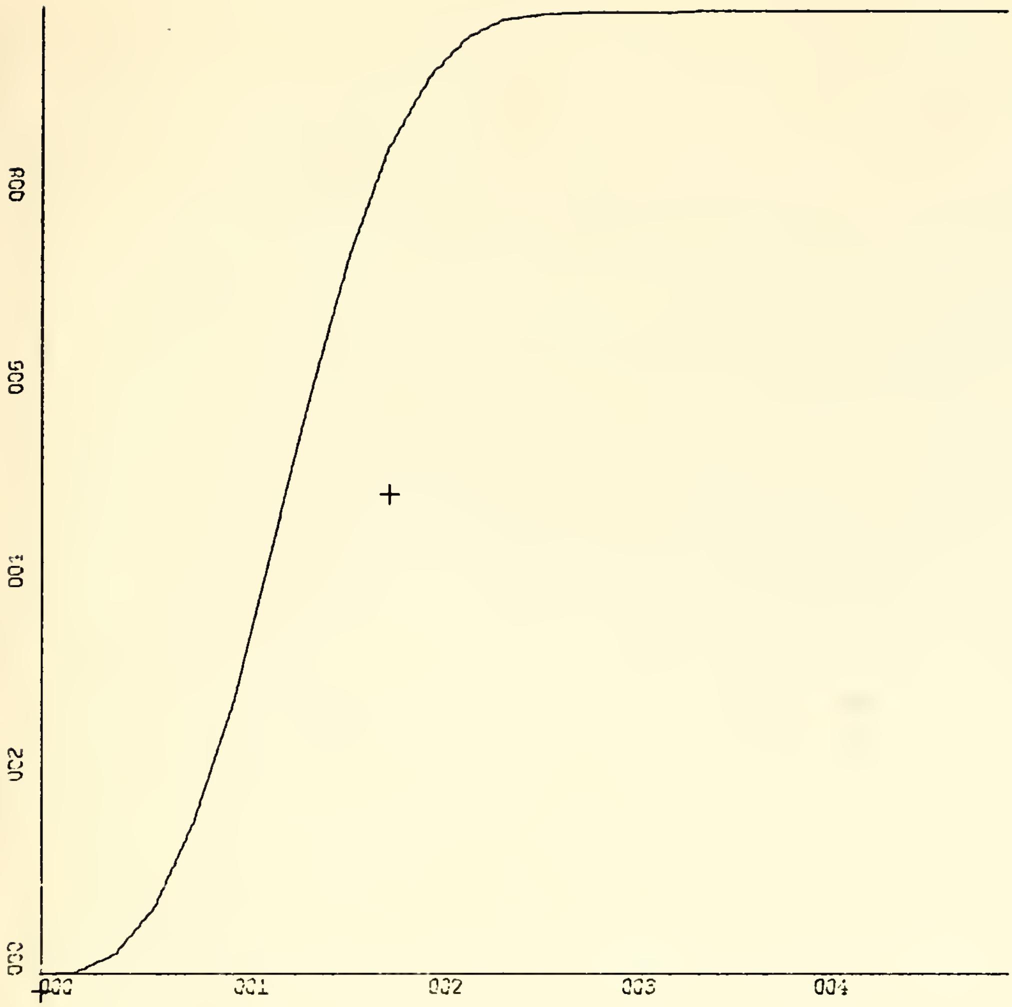
FIGURE 128



FOR DELTA T = 1.6    TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01138 |
| 1.6 | 0.44794 |
| 3.2 | 0.98867 |

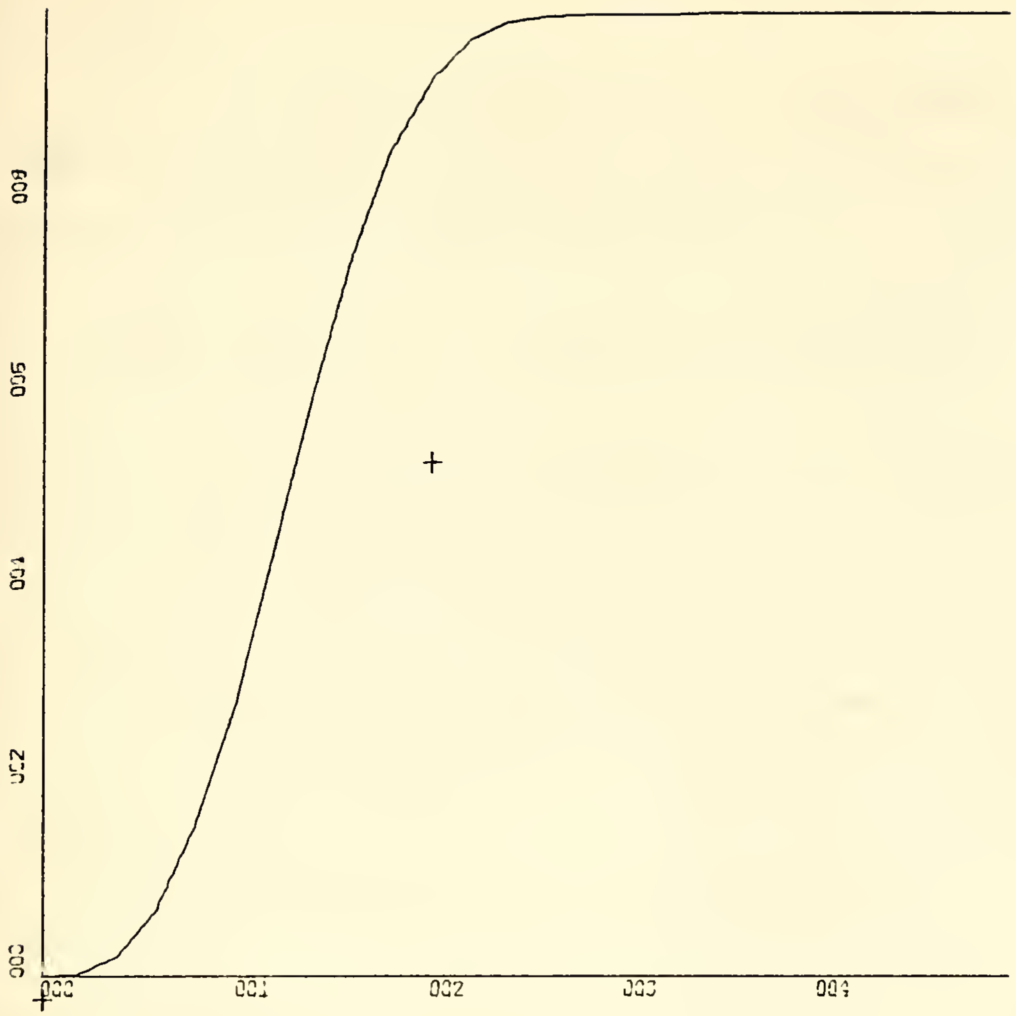
FIGURE 129



FOR DELTA T = 1.8   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.01620 |
| 1.8 | 0.49642 |

FIGURE 130



FOR DELTA T = 2.0   TFIN = 5.0

| T | Q(N) |
|---|---|
| 0.0 | -0.02222 |
| 2.0 | 0.53333 |

FIGURE 131

E. CASE 10

No one of the previous cases has a time varying coefficient which is exponential. So, from Ref. 2 this case was extracted. The equation is:

$$C \frac{de}{dt} + \frac{e}{r(t)} = 0 \qquad \text{being } r(t) = R \exp\left(\frac{t}{T_1}\right)$$

where $T_1$ is a factor in the variation of r. Dividing all the terms by C yields:

$$\frac{de}{dt} + \frac{1}{T_2} \exp\left(\frac{-t}{T_1}\right) e = 0$$

where $T_2$ = RC.

For this example $T_1$ and $T_2$ were made equal to one and $e_{(t=0)} = 1.0$ . Rewriting the equation and expressing its Laplace transform in powers of $s^{-1}$:

$$\dot{e} + Ce = 0 \qquad C = e^{-t}$$

$$sE - 1 + CE = 0$$

$$E = \frac{1}{s + C} = \frac{1}{s} - \frac{Cs^{-2}}{1 + Cs^{-1}}$$

If the rules in Example 2, Section IIB were followed, substituting the z-forms of $s^{-k}$, dividing by T and adding the final results of the quotient to 1.0 which is the time function for 1/s:

$$E_A^* = \frac{-C\,\frac{T^2}{12}\,\frac{1 + 10z^{-1} + z^{-2}}{(1 - z^{-1})^2}}{1 + C\,\frac{T}{2}\,\frac{1 + z^{-1}}{1 - z^{-1}}}$$

For T = 0.1

$$E_A^* = \frac{-0.1 - z^{-1} - 0.1z^{-2}}{(12/C + 0.6) - (24/C)z^{-1} + (12/C - 0.6)z^{-2}}$$

Dividing the denominator into the numerator and adding one to the result one gets

$$0.99206 + 0.91412z^{-1} + 0.85007z^{-2} + \cdots$$

However, looking at the graph (Fig. 132) the precise and the approximated solution start diverging after a while. This is due to the fact that in

$$E = \frac{1}{s + C}$$

if C goes to zero the equation becomes

$$E = 1/s$$

which is the final value that the approximated solution approaches.

This fact is very important because it leads to these important conclusions:

1. If $C_1$ goes to zero with time, first check the Laplace transform and the initial equation substituting $C_1$ by zero. If by inspection they do not lead to the same type of solution, this method does not apply.

2. Verify, before doing the division if any of the varying coefficients, unexpectedly, goes to infinity; by dividing by zero, for instance.

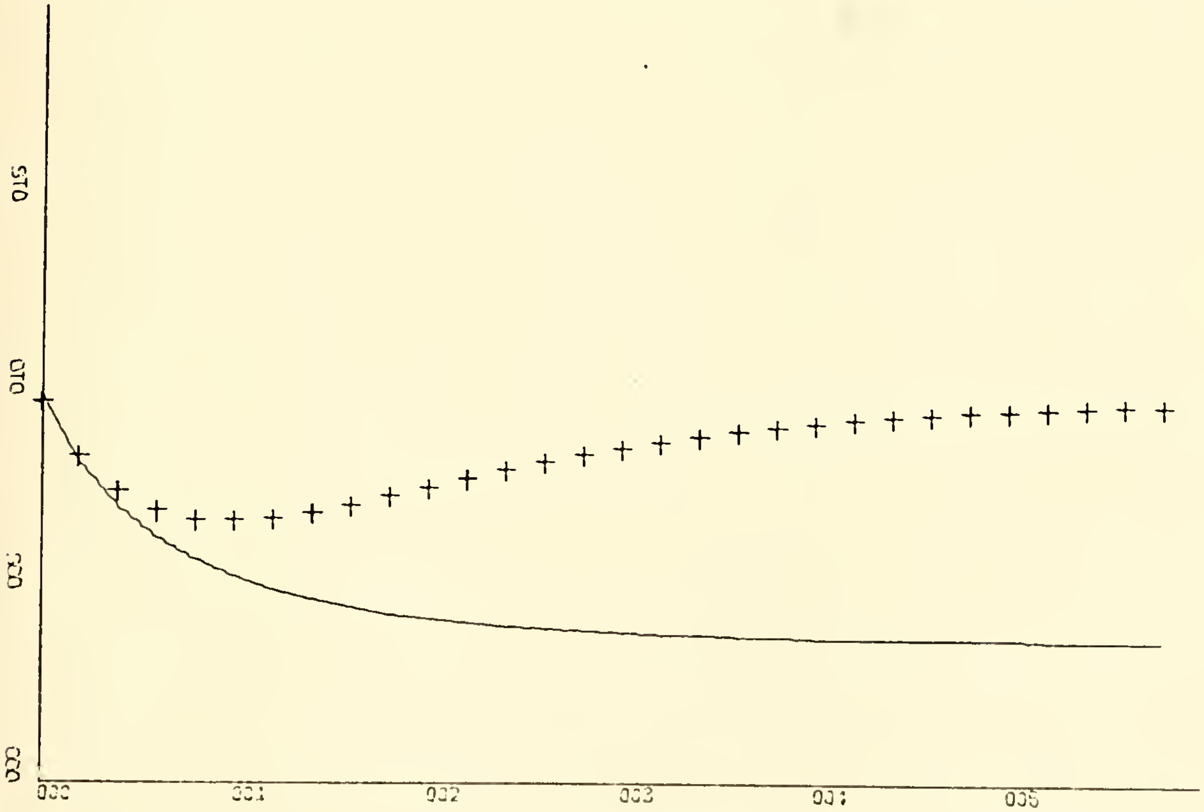


Figure 132

X-scale = 1.0 units/inch
Y-scale = .5 units/inch

— Precise solution
+ Approximate solution

## IV. CONCLUSIONS

This technique, for the successfully area covered in this paper (linear differential equations with time varying coefficients), showed to have a good precision even for relatively large iteration steps. The labor necessary to complete a problem by handwork is not very involved and needs only a small algorithm for computer solution with C.P.U. times much smaller than existing numerical methods.

However, with time varying coefficients that go to zero with time, attention must be paid to the application of this method. As a general rule, this method does not apply.

In some cases, the final equation for the approximate solution, before the division is performed, must be altered in order to not have time varying coefficients (referred as $C_i$ in Section IIC) in the numerator. Caution is necessary to verify, before starting the division steps if any varying coefficient unexpectedly has a value of infinity; by division by zero when substituting a particular value of $C_i$.

The technique tested in this paper does not apply to the case of linear differential equations with varying coefficients only. Linear differential equations with constant coefficients and nonlinear differential equations are, as well, solved by this technique.

These areas, in the opinion of the author, are valuable as topics for other theses.

## LIST OF REFERENCES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